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Correlation functions for M^N/S_N orbifolds

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Abstract

We develop a method for computing correlation functions of twist operators in the bosonic 2-d CFT arising from orbifolds M^N/S_N, where M is an arbitrary manifold. The path integral with twist operators is replaced by a path integral on a covering space with no operator insertions. Thus, even though the CFT is defined on the sphere, the correlators are expressed in terms of partition functions on Riemann surfaces with a finite range of genus g. For large N, this genus expansion coincides with a 1/N expansion. The contribution from the covering space of genus zero is 'universal' in the sense that it depends only on the central charge of the CFT. For 3-point functions we give an explicit form for the contribution from the sphere, and for the 4-point function we do an example which has genus zero and genus one contributions. The condition for the genus zero contribution to the 3-point functions to be non–vanishing is similar to the fusion rules for an SU(2) WZW model. We observe that the 3-point coupling becomes small compared to its large N limit when the orders of the twist operators become comparable to the square root of N - this is a manifestation of the stringy exclusion principle.

1 Introduction

The AdS/CFT correspondence gives a remarkable relation between string theory on a spacetime and a certain conformal field theory (CFT) on the boundary of this spacetime [1]. In particular the near horizon geometry of D3 branes gives the space $AdS_5 \times S^5$, and string theory on this space is conjectured to be dual to N=4 supersymmetric Yang-Mills on the boundary of the AdS_5. When the string theory is weakly coupled, tree level supergravity is a valid low energy approximation. The dual CFT at this point is a strongly coupled Yang-Mills theory, which cannot therefore be studied perturbatively. On the other hand weakly coupled Yang-Mills theory is dual to string theory in a domain of parameters where the latter cannot be approximated by supergravity on a gently curved spacetime. In spite of this fact, it turns out that certain quantities computed in the supergravity limit of string theory agree with their corresponding dual quantities in the Yang-Mills theory, where the latter computation is done at *weak* coupling. One believes that such an agreement is due to the supersymmetry which is present in the theory; this supersymmetry would for example protect the dimensions of chiral operators from changing when the coupling is varied. Interestingly, the values of 3-point correlation functions of chiral operators are also found to agree, when we compare the tree level supergravity calculation on AdS space with the computation in the free Yang-Mills theory [2]; the latter is just the result obtained by Wick contractions among the fields in the chiral operators. It is not clear if the 3-point function of chiral operators is protected against change of coupling at all values of N; the above result just tells us that the large N results agree between the weak and strong coupling limits.[1]

One is thus led to ask: are the 3-point functions of chiral operators protected in the other cases of the AdS/CFT correspondence? In particular we will be interested in the case of the D1-D5 system [1, 4], which gives a near-horizon geometry $AdS_3 \times S^3 \times M$, where M is a torus T^4 or a $K3$ space. This system is of great interest for the issues related to black holes, since it yields, upon addition of momentum excitations, a supersymmetric configuration which has a classical (i.e. not Planck size) horizon. In particular, the Bekenstein entropy computed from the classical horizon area agrees with the count of microstates for the extremal and near extremal black holes [5]. Further, the low energy Hawking radiation from the hole can be understood in

[1] See however [3] for an analysis of the finite N case.

terms of a unitary microscopic process, not only qualitatively but also quantitatively, since one finds an agreement of spin dependence and radiation rates between the semiclassically computed radiation and the microscopic calculation [6]. While it is possible to use simple models for the low energy dynamics of the D1-D5 system when one is computing the coupling to massless modes of the supergravity theory, it is believed that the exact description of this CFT must be in terms of a sigma model with target space being a deformation of the orbifold M^N/S_N, which is the symmetric orbifold of N copies of M. (Here $N = n_1 n_5$, with n_1 being the number of D1 branes and n_5 being the number of D5 branes, and we must take the low energy limit of the sigma model to obtain the desired CFT.) In particular we may consider the 'orbifold point' where the target space is exactly the orbifold M^N/S_N with no deformation. It was argued in [7] that this CFT does correspond to a certain point in the moduli space of string theories on $AdS_3 \times S^3 \times M$, but at this point the string theory is in a strongly coupled domain where it cannot be approximated by tree level supergravity on a smooth background. The orbifold point is the closest we can get to a 'free' theory on the CFT side, and thus this point is the analogue of free N=4 supersymmetric Yang-Mills in the D3 brane example. Thus one would like to compare the three point functions of chiral operators in the supergravity limit with the 3-point functions at the orbifold point, to see if we have an analogue of the surprising agreement that was found in the case of the AdS_5 - 4-d Yang-Mills duality.

The orbifold group in our case is S_N, the permutation group of N elements. This group is nonabelian, in contrast to the cyclic group Z_N which has been studied more extensively in the past for computation of correlation functions in orbifold theories [9]. Though there are some results in the literature for general orbifolds [10], the study of nonabelian orbifolds is much less developed than for abelian orbifolds.

It turns out however that the case of the S_N orbifolds has its own set of simplifications which make it possible to develop a technique for computation of correlation functions for these theories. The essential quantities that we wish to compute are the correlation functions of 'twist operators', in the CFT that arises from the infra-red limit of the 2-d sigma model with target space M^N/S_N. If we circle the insertion of a twist operator of the permutation group S_N, different copies of the target space M permute into each other. We pass to the covering space of the 2-d base space, such that on this covering space the fields of the CFT are single-valued. For the special case where the orbifold group is S_N, the path integral on the base space with

twist operators inserted becomes a path integral over the covering space for the CFT with only one copy of M, with no operator insertions. Thus the correlation functions of twist operators can be rewritten as partition functions on Riemann surfaces of different genus for the CFT arising from one copy of M.

In the simplest case, which is also the case giving the leading contribution at large N, the genus of the covering surface is zero, and we just get the partition function on the sphere. But the metric on this sphere is determined by the orders and locations of the twist operators. We can write this metric as $g = e^{\phi}\hat{g}$, for some fiducial metric $\hat{g}$, provided we take into account the conformal anomaly given by the Liouville action for ϕ. It turns out that ϕ is harmonic outside a finite number of isolated points, so the Liouville action can be computed by observing the local behavior of the covering surface at these points. In this manner we can compute any correlation function of twist operators. For given operators we get contributions from only a finite range of genera for the covering surface.

We compute the 2-point function of twist operators, from which we recover their well known scaling dimensions. We then compute the contribution to 3-point functions which comes from covering surfaces of genus zero. This gives the complete result for the fusion coefficients of twist operators for a subset of cases (the 'single overlap' cases) and the leading result at large N for all cases. We then compute the 4-point function for twist operators of order 2; this correlator has contributions from genus 0 and genus 1, and we compute both contributions.

We find a certain 'universality' in the correlation functions, since the Liouville action depends only on the central charge c of the CFT with target space M. The contribution from higher genus covering surfaces will involve the partition functions at those genera, while the leading order result coming from the covering surface of genus zero will depend only on c. These observations generalize the fact that the dimensions of the twist operators depend only on c.

We address only bosonic operators in this paper, though we expect the extension to the supersymmetric case to be relatively straightforward. Thus we also do not address the comparison to supergravity; this comparison should be carried out only for the supersymmetric case. But we do observe some features of our correlation functions that accord with some of the patterns that emerge in the supergravity computation. In particular we find a similarity between the condition for the 3-point correlator to have a contribution

from genus zero covering surfaces, and the condition for primaries to fuse together in the SU(2) Wess-Zumino-Witten model.

There are several earlier works that relate to the problem we are studying, in particular [8, 11, 12, 14, 15, 16]. We will mention these in more detail in the context where they appear.

The plan of this paper is the following. Section 2 describes our method of computing correlation functions. In section 3 we compute the 2-point functions, and thus recovers the scaling dimensions of the twist operators. In section 4 we construct the map to the covering space for 3-point functions, for the case where the covering space is a sphere. In section 5 we find the Liouville action associated to this map. In section 6 we use the above results to obtain the contribution to 3-point functions from covering surfaces of genus zero. Section 7 discusses an example of a 4-point function. Section 8 is a discussion.

2 Computing correlation functions through the Liouville action

2.1 The twist operators

Let us consider for simplicity the case $M = R$, i.e. the noncompact real line. Then the sigma model target space M^N/S_N can be described through a collection of N free bosons $X^1, X^2, \dots X^N$, living on a plane parameterized by the complex coordinate z. We will see later that we can extend the analysis directly to other CFTs. Let the z plane have the flat metric

$$ds^2 = dzd\bar{z}. \tag{2.1}$$

The CFT is defined through a path integral over the values of the X^i, with action

$$S = \int d^2z \; 2\partial_z X^i \partial_{\bar{z}} X^i. \tag{2.2}$$

We now make the definition of this partition function more precise. We cut off the z plane at a large radius

$$|z| = \frac{1}{\delta}, \quad \delta \;\text{ small.} \tag{2.3}$$

We want boundary conditions at this large circle to represent the fact that the identity operator has been inserted at infinity. We will explain the norm of this boundary state later on.

Thus we imagine that our CFT is defined on a 'sample' of size $1/\delta$ – correlation functions are to be computed by putting the operators at $|z_i| << 1/\delta$ and we take δ to zero at the end of the calculation. (An exception will be the insertion of an operator at infinity, which we will have to define separately.) We write the path integral for a single boson on the z plane as

$$Z_\delta = \int DX\ e^{-S}. \tag{2.4}$$

The path integral for N bosons, with no twist operator insertions, is $(Z_\delta)^N$.

The twist operator $\sigma_{12}(z_1)$ can be described through the following. Cut a circular hole of radius ϵ in the z plane about the point z_1. While the path integral over $X^i, i = 3, \ldots N$ is left unchanged, we modify the boundary conditions on X^1, X^2 such that as we go around the hole at z_1 we get

$$X^1 \to X^2, \quad X^2 \to X^1. \tag{2.5}$$

We call this operator $\sigma^\epsilon_{12}(z_1)$, where the ϵ in the superscript reminds us of the regulation used to define the twist. Note that we still have to define more precisely the state that is inserted at the edge of the hole of radius ϵ - we will do this below. If we want to maintain the boundary condition at the circle $|z| = 1/\delta$ (and introduce no twist there) we must insert at some point z_2 another such twist operator σ_{12}; the size of the circular hole around z_2 is also ϵ. Let us compute the path integral with these boundary conditions, and call it $Z_{\epsilon,\delta}[\sigma_2(z_1), \sigma_2(z_2)]$. Then we define the correlation function

$$\langle \sigma^\epsilon(z_1)\sigma^\epsilon(z_2)\rangle_\delta \equiv \frac{Z_{\epsilon,\delta}[\sigma_2(z_1), \sigma_2(z_2)]}{(Z_\delta)^N}. \tag{2.6}$$

We will later define rescaled twist operators, and the cutoffs ϵ, δ will disappear from all final answers.

2.2 Path integral on the covering space

The functions X^1, X^2 above were not single valued in the z plane due to the insertion of the twist operators. We wish to pass to a covering space where these functions would become one single valued function. Since the fields

$X^3, \ldots X^N$ are not involved in the twist, they cancel out in the RHS of (2.6). We thus consider only X^1, X^2 in the following.

Consider a configuration of the fields X^1, X^2 which contributes to the path integral. Consider a simply connected patch of the z plane, which excludes the holes around z_1, z_2. Over this patch there are two functions defined: one for each field, though due to the twists there is no global way to label the functions uniquely as being X^1 or X^2. Take any one of the functions over this patch - call it $X(z)$. To construct the covering surface Σ, let this open set in z be a patch on Σ, with the complex structure given by z, and the metric also equal to the metric (2.1) of the z plane. There is one field X that will be defined over Σ, and over this patch let it be the above mentioned function $X(z)$. Now consider another such simply connected open patch, partly overlapping with the first, and use it to define another patch on Σ. Clearly, as we go around the point z_1, following these overlapping patches, the surface Σ will look locally like the Riemann surface of a function

$$t = (z - z_1)^{1/2}. \tag{2.7}$$

Further, the functions X^1, X^2 will be both encoded in the function X which will be single valued on Σ, and the action of the configuration of X^1, X^2 will be reproduced if in each patch we use for X the action

$$S = \int_{\text{patch}} d^2z \; 2\partial_z X \partial_{\bar{z}} X. \tag{2.8}$$

The coordinate z cannot be a globally well defined coordinate for Σ, since a generic value of z corresponds to two points on Σ. We will call our choice of coordinate on Σ as t, which will be locally holomorphically related to z.

In (2.8) we must evaluate the path integral on the patch using the metric induced from the z plane on the patch; the path integral depends in general on the metric and the physical problem is defined through the metric chosen on the z plane. Thus in terms of the coordinate t used to describe Σ we will have

$$ds^2 = dzd\bar{z} = |\frac{dz}{dt}|^2 dt d\bar{t}. \tag{2.9}$$

For the example above we can take

$$z = a\frac{t^2}{2t-1}. \tag{2.10}$$

Near the location $z = 0, t = 0$, we evidently have $z \sim t^2$, so that t parameterizes the covering surface near $z = 0$. But the map between z and t is singular also at $z = a, t = 1$, since near $t = 1$

$$\frac{dz}{dt} \approx 2a(t-1), \qquad z - a \approx a(t-1)^2. \tag{2.11}$$

Thus the twist operators are located at $z_1 = 0$ and $z_2 = a$.

The covering space Σ (parameterized by t) is a double cover of the original sphere parameterized by z. We had cut the z plane at $|z| = 1/\delta$, and inserted the identity there. This boundary in the z space corresponds to two regions in the t space: $z \to \infty$ maps to $t \to \infty$ as well as $t \to 1/2$. Thus in the t plane we will have a boundary near infinity, as well as in a small disc cut out around $t = 1/2$. We will have the identity inserted at each of these boundaries; the precise state with norm will be defined in the subsection below.

We now note that we have simply a path integral over a single free boson X on Σ - there is no twist operator left in the problem, and any boundaries present on Σ carry only the identity state. We will show below how to close these holes in Σ, and then we will have just a path integral over a closed surface to compute.

The method of passing to a covering space to analyze orbifold correlation functions has been studied by many authors, for example [8, 9]. The observation that for symmetric orbifolds one gets a single copy of the target space with no nontrivial operator insertions on the covering space is implicit in [11]. The notion of passing to the covering space to take into account the twist operators for symmetric orbifolds is also used in the computation of partition functions in [12]. The Z_2 orbifold is the same as the S_2 orbifold, and the map to the covering space was used in [8, 9] to find the 4-point correlation of twist operators for the Z_2 orbifold of a complex boson.

We will depart from the usual way of computing the correlation functions of twist operators, and use a different way which we describe below. The usual computation for $\langle\sigma\sigma\rangle$ (and the one adopted in [9, 11]) proceeds by first finding

$$f \equiv \frac{\langle \partial X^i(z) \partial X^i(w) \sigma(z_1) \sigma(z_2) \rangle}{\langle \sigma(z_1) \sigma(z_2) \rangle} \tag{2.12}$$

by looking at the singularities of f as a function of z, w, and constructing a function with these singularities. One then takes the limit $z \to w$, subtracts the singularity and constructs the stress tensor $T = \frac{1}{2}\partial X^i \partial X^i(w)$. Next, one

uses the conformal Ward identity to relate $\langle T\sigma\sigma\rangle$ to $\langle\partial\sigma\sigma\rangle$, thus obtaining an expression for $\partial_z \log\langle\sigma(z)\sigma(0)\rangle$. Solving this equation gives the functional form of the 2-point function, and the dimension of σ can be read off from the solution.

A similar analysis can be done for the 3-point function, but the functional form of the 3-point function of primary fields is determined by their dimensions, and would tell us nothing new. One cannot find the fusion coefficients C_{ijk} between the twist operators from the 3-point analysis because the method does not determine the overall normalization of the correlator. Thus to find the C_{ijk} one applies the method to the 4-point function $\langle\sigma\sigma\sigma\sigma\rangle$, finds the functional form of this correlator, and then uses factorization to extract the C_{ijk}.

To be able to use such a method one must have a simple stress tensor which can be written as a product of fields, each of which has a simple known behavior near the twist operators. One must also use inspection to construct the correlators like (2.12). Further, to find the C_{ijk} we need to go up to the 4-point function.

The method we use will apply to S_N orbifolds, but not for example to Z_N orbifolds with $N > 2$. On the other hand we will not need that the stress tensor have a simple form (in fact we will not use the stress tensor at all). Further, we can compute the C_{ijk} using only the 2 and 3-point functions. Thus the method is suited to the computation of correlation functions for CFTs arising from sigma models with target space M^N/S_N, which arise in D-brae physics. The method also brings out the fact that many quantities for symmetric orbifolds are 'universal' in the sense that they do not depend on the details of the the manifold M.

2.3 Closing the punctures

For the discussion below the order and number of twist operators can be arbitrary, but for explicitness we assume that the covering space Σ has the topology of a sphere, and we use the correlator $\langle\sigma_2\sigma_2\rangle$ as an illustrative example. It will be evident that no new issues arise for other correlators or when Σ has higher genus; we will mention the changes for higher genus where relevant.

As it stands the covering surface Σ that we have constructed has several 'holes' in it. We will now give a prescription for closing these holes, thus making a closed surface which we also call Σ. The prescription for closing

the holes will amount to defining precisely the states to be inserted at various boundaries. If the surface is closed then we can use the Liouville action to find the path integral after change of metric; on an open surface the boundary states can change as well.

The holes are of the following kinds:

(i) The holes in the finite z plane at the insertion of the twist operators. These holes are circles with radius ϵ in the z plane, and lift to holes in Σ under the map $t(z)$.

(ii) The holes in Σ at finite values of t, arising from the fact that we have cut the z space at $|z| = 1/\delta$. These holes are located at points t_0 where the map behaves as $z \sim \frac{1}{t-t_0}$.

(iii) The hole in Σ at $t = \infty$, which also arises from the fact that the z space is cut at $|z| = 1/\delta$. If there is no twist operator at $z = \infty$, then we have $z \sim t$ for large t, and the hole in Σ is the image of $|z| = 1/\delta$ under this map.

We first complete our definition of the twist operators by defining the state inserted at the edge of the cut out hole; this addresses holes of type (i) above. We had said above that the twist operator σ_{12} imposes the boundary condition (2.5), but this does not specify the operator completely. In fact there are an infinite number of operators, with increasing dimensions, which all create the same twist and in general create some further excitation of the fields. We define σ_{12} to be the operator from this family with lowest dimension. After mapping the problem to the t space, we need to ask what operator insertions at the points $t = 0$, $t = 1$ give the slowest power law fall off for the partition function when the distance a between the twist operators is increased. The answer is of course that we must insert a multiple of the identity operator at the punctures in the t space.

But we will also need to know the norm of this state, and would thus like to construct it through a path integral. Thus let the covering surface be locally defined through (2.7). For $|t| > \epsilon^{1/2}$ the metric on the covering surface is the one induced from the z plane

$$ds^2 = dzd\bar{z} = dtd\bar{t}|\frac{dz}{dt}|^2 = 4|t|^2 dtd\bar{t}, \qquad |t| > \epsilon^{1/2}. \tag{2.13}$$

We 'close the hole' in the t space by choosing, for $|t| < \epsilon^{1/2}$, the metric

$$ds^2 = 4\epsilon dtd\tilde{t}, \quad |t| < \epsilon^{1/2}. \tag{2.14}$$

Thus we have glued in a 'flat patch' in the t space to close the hole created in the definition of the twist operator. The metric is continuous across the boundary $|t| = \epsilon^{1/2}$, but there is curvature concentrated along this boundary. The path integral of X over the disc $|t| < \epsilon^{1/2}$ creates the required state along the edge of the hole. The map (2.7) is only the leading order approximation to the actual map in general, but our prescription is to 'close with a flat patch' the hole in the t space, where the hole is the image on Σ of a circular hole in the z plane. As $\epsilon \to 0$, the small departure of the map from the form (2.7) will cease to matter.

Note that we could have chosen a different metric to replace the choice (2.14) inside the hole, but this would just correspond to a different overall normalization of the twist operator. (Thus it would be like taking a different choice of ϵ.) Once we make the choice (2.14) then we must use the same construction of the twist operator in all correlators, and then the non–universal choices in the definitions will cancel out.

The other holes, of types (ii) and (iii), arise from the hole at infinity in the z plane, and we proceed by first replacing the z plane by a closed surface. We take another disc with radius $1/\delta$ (parameterized by a coordinate $\tilde{z}$) and glue it to the boundary of the z plane. Thus we get a sphere with metric given by

$$\begin{aligned} ds^2 &= dz d\bar{z}, \quad |z| < \frac{1}{\delta}, \\ &= d\tilde{z} d\bar{\tilde{z}}, \quad |\tilde{z}| < \frac{1}{\delta}, \\ \tilde{z} &= \frac{1}{\delta^2}\frac{1}{z}. \end{aligned} \tag{2.15}$$

The path integral over the second disc defines a state at the boundary of the first disc. This state is proportional to the identity. But further, our explicit construction gives the state a known norm, which is something we needed to completely define the path integrals like (2.4) and (2.6).

Since the z space is closed at infinity, we find that the holes of type (ii) and (iii) are now automatically closed in Σ – since we make Σ as a cover of the z sphere with metric on every patch induced from the metric on this z sphere.

The space Σ is now a closed surface with a certain metric, and the path integral giving $Z_{\epsilon,\delta}[\sigma_2(z_1), \sigma_2(z_2)]$ in (2.6) is to be carried out on this closed surface.

2.4 The method of calculation

We had found above that if we have twist operators σ_{12} at $z=0$, $z=a$, then the partition function with the twist operators inserted equals that for a single field X on the double cover Σ of the z plane given by the map (2.10). Σ is also a sphere with infinitesimal holes cut out, but since only the identity operator is inserted at these punctures, we can close the holes and just get the partition function of X on a closed surface Σ.

At this point one might wonder that since this partition function is some given number, how does it depend on the parameter a, which gave the separation between the twist operators in the z plane? The point is that even though Σ is a sphere for all a, the *metric* on this sphere depends on a – this is evident from (2.9).

We *can* compute the partition function of X on Σ using some fixed fiducial metric $\hat{g}$ on the t space, but we must then take into account the conformal anomaly, which says that if $ds^2 = e^{\phi} d\hat{s}^2$, then the partition function $Z^{(s)}$ computed with the metric ds^2 is related to the partition function $Z^{(\hat{s})}$ computed with $d\hat{s}^2$ through

$$Z^{(s)} = e^{S_L} Z^{(\hat{s})}, \tag{2.16}$$

where

$$S_L = \frac{c}{96\pi} \int d^2t \sqrt{-g^{(\hat{s})}} [\partial_\mu \phi \partial_\nu \phi g^{(\hat{s})\mu\nu} + 2R^{(\hat{s})} \phi] \tag{2.17}$$

is the Liouville action [13]. Here c is the central charge of the CFT. Since we are considering the theory of a single free field X on Σ, we have $c=1$.

Let us choose the fiducial metric $\hat{g}$ on Σ to be (in the case where Σ is a sphere)

$$\begin{aligned} d\hat{s}^2 &= dt d\bar{t}, \quad |t| < \frac{1}{\delta'} \\ &= d\tilde{t} d\bar{\tilde{t}}, \quad |\tilde{t}| < \frac{1}{\delta'}, \\ \tilde{t} &= \frac{1}{\delta'^2} \frac{1}{t}. \end{aligned} \tag{2.18}$$

We will let $\delta' \to 0$ at the end. Thus we have chosen the fiducial metric on the t space to be the flat metric of a plane up to a large radius $1/\delta'$, after which we glue an identical disc at the boundary to obtain the topology of a sphere, just as we did for the z space.

From (2.16), (2.17) we see that if we increase ϕ by a constant, then Z changes by a known factor. Using that fact that for a sphere $\int \sqrt{g}R = 8\pi$, we find that the partition function of X on the sphere with metric (2.18) is

$$Z_{\delta'} = Q\ (\delta')^{-\frac{c}{3}} = Q\ (\delta')^{-\frac{1}{3}}. \tag{2.19}$$

Thus we will have $Z^{(\hat{s})} = Z_{\delta'}$. Here Q is a constant that is regularization dependent and cannot determined by anything that we have chosen so far. (Q determines the size of the sphere for which the partition function will attain the value unity; since the CFT has no inbuilt scale we cannot find the value of this size in any absolute way.) Q will cancel out in all final calculations.

The partition function of one boson on the z sphere with the metric (2.15) is Z_δ (cf. eq. (2.4)), and we have

$$Z_\delta = Q\delta^{-\frac{c}{3}} = Q\delta^{-\frac{1}{3}}. \tag{2.20}$$

2.5 Contributions to S_L

The partition function with twist operators inserted can be written as

$$Z_{\epsilon,\delta}[\sigma_{n_1}(z_1), \ldots, \sigma_{n_k}(z_k)] = e^{S_L} Z^{(\hat{s})}. \tag{2.21}$$

Thus the computation of the correlation function boils down to computing S_L. There are three types of contributions to S_L, which we will analyze separately

$$S_L = S_L^{(1)} + S_L^{(2)} + S_L^{(3)}, \tag{2.22}$$

$S_L^{(1)}$ will give the essential numerical contributions to the correlation functions (as well as regulation dependent quantities), while $S_L^{(2)}$ and $S_L^{(3)}$ give only regulation dependent quantities; regulation parameters cancels out at the end.

(a) We have cut out various discs from the z plane where the physical theory is defined: we have removed infinity by taking $|z| < 1/\delta$ and have also cut out circles of radius ϵ around the twist operator insertions. Let us call this region of z the 'regular region'. This 'regular' region of the z space has an image in the t space, which we call the 'regular region' on Σ. On Σ we will find, apart from the obvious cuts around the images of the twist operators and a cut near $|t| = \infty$, further possible cuts around images of

$z = \infty$ as discussed in subsection 2.3. Let the contribution to S_L from this 'regular region' of Σ be called $S_L^{(1)}$.

To evaluate (2.21) we need to choose a fiducial metric on the t space. Suppose that the map $z(t)$ has the form $z \sim bt$ as $t \to \infty$. (When there is no twist operator at infinity the map can be taken to have this form.) Let this fiducial metric $d\hat{s}^2$ be of the form (2.18) with

$$\frac{1}{\delta} < \frac{b}{\delta'}. \tag{2.23}$$

With this choice the boundary $|z| = 1/\delta$ gets mapped to a curve inside the disc $|t| < 1/\delta'$ (i.e. into the 'first half' of the t sphere).

In this 'regular region' of Σ, the fiducial metric (2.18) is flat, and so there is no contribution from the $R\phi$ term in (2.17). Thus we have

$$S_L^{(1)} = \frac{1}{96\pi} \int d^2t[\partial_\mu\phi\partial^\mu\phi], \tag{2.24}$$

where the integral extends over the region described above. We rewrite (2.24) as

$$S_L = -\frac{1}{96\pi} \int d^2t[\phi\partial_\mu\partial^\mu\phi] + \frac{1}{96\pi} \int_\partial \phi\partial_n\phi. \tag{2.25}$$

Here ∂ is the boundary of the 'regular region' of Σ, and ∂_n is the normal derivative at the boundary. From (2.9) we find that

$$\phi = \log[\frac{dz}{dt}] + \log[\frac{d\bar{z}}{d\bar{t}}], \tag{2.26}$$

so that

$$\partial_\mu\partial^\mu\phi = 4\partial_t\partial_{\bar{t}}\phi = 0, \tag{2.27}$$

and we get

$$S_L = \frac{1}{96\pi} \int_\partial \phi\partial_n\phi. \tag{2.28}$$

The boundaries of the 'regular region' are of two kinds: those arising from the holes of size $|z - z_i| = \epsilon$ cut around the twist operators, and those arising from the cutoff at infinity ($|z| = 1/\delta$). Consider the boundary of the hole arising from some twist operator $\sigma_n(z_i)$. We regulated the twist operator by choosing this hole to be a circle in the z plane, so we start by looking at a segment of the boundary using the coordinate z. We have

$$\partial_n = -\frac{1}{|z|}(z\partial_z + \bar{z}\partial_{\bar{z}}). \tag{2.29}$$

Writing $z = |z|e^{i\theta}$, one finds

$$\int ds = |z| \int d\theta = \frac{|z|}{i} \int \frac{dz}{z} = \frac{|z|}{-i} \int \frac{d\bar{z}}{\bar{z}}. \tag{2.30}$$

Thus we get

$$\int_{\partial} ds \ \phi \partial_n \phi = i \int dz \ \phi \partial_z \phi \ + \ c.c. \tag{2.31}$$

Since z is holomorphically dependent on t, we can write

$$dz \ \partial_z \phi = dt \ \partial_t \phi. \tag{2.32}$$

We can thus write for the contribution to S_L from any hole

$$\frac{1}{96\pi} \int_{\partial} ds \ \phi \partial_n \phi = \frac{1}{96\pi} [i \int dt \ \phi \partial_t \phi \ + \ c.c.], \tag{2.33}$$

where ϕ is given through (2.26). A similar analysis applies to all the other boundaries of the 'regular region' on Σ, and we compute (2.33) for each such boundary. Since the 'holes' on Σ are infinitesimal size punctures, computing (2.33) needs only the leading order behavior of ϕ at the punctures.

(b) We had cut out holes of radius ϵ in the z plane around the insertions of the twist operators, and these gave corresponding holes in the 'regular region' of Σ. We now compute the contribution to S_L from the part H of Σ that is used to close such a hole. Since we had closed these holes with the flat metric (2.14), and since the fiducial metric we use on Σ is also flat in H ($d\hat{s}^2 = dt d\bar{t}$), we get $\phi =$ constant, and so there is no contribution from the kinetic term in (2.17). Note that at the boundary of H we have $\partial_t \phi$ nonzero but bounded, then since the area of this boundary is zero (the boundary is one-dimensional) we get no contribution to the kinetic term from the boundary either. The curvature term in (2.17) is zero, since the curvature of the fiducial metric is zero throughout the region where the twist operators are inserted. Thus we get no contribution to S_L from these regions H of Σ.

(c) Now consider the contributions from the points that have finite t, but $z \to \infty$. The 'regular region' on Σ had excluded the image of $|z| > 1/\delta$. This image will have a small disc D around some finite t_0, if we have

$$z \approx \frac{\alpha}{t - t_0} + \beta + \ldots \tag{2.34}$$

The fiducial metric we are using on Σ is flat here, so there is no contribution from the curvature term in (2.17). The region inside the disc D has a metric

induced from the 'second half' of the z sphere (i.e. the part parameterized by $\tilde{z}$ in (2.15)) so that the metric is $ds^2 = d\tilde{z}d\bar{\tilde{z}}$. Thus

$$\begin{aligned}
\tilde{z} &= \frac{1}{\delta^2}\frac{1}{z} \approx \frac{1}{\delta^2\alpha}(t-t_0) - \frac{\beta}{\delta^2\alpha^2}(t-t_0)^2, \\
\phi &= \log\frac{d\tilde{z}}{dt} + c.c. \approx \log\frac{1}{\delta^2\alpha} - \frac{2\beta}{\alpha}(t-t_0) + c.c., \\
\partial_t\phi &\approx -\frac{2\beta}{\alpha}.
\end{aligned} \tag{2.35}$$

The area of the disc D in the fiducial metric is $\pi|t-t_0|^2 \approx \pi(|\alpha|\delta)^2$. As $\delta \to 0$, we find that $\int d^2t\ \partial_t\phi\partial_{\bar{t}}\phi \to 0$. Thus we get no contribution to S_L from these images of the cut at infinity.

(d) Now we look at the region of Σ near $t=\infty$. Let $z \approx bt$ for large t. Let the image of $|z| = 1/\delta$ be the contour C on Σ. By the choice (2.23) and the fact that C satisfies $|t| \approx \frac{1}{b\delta}$, we find that C is inside the curve $|t| = 1/\delta'$. Let the contribution to S_L from the region between C and $|t| = 1/\delta'$ be called $S_L^{(2)}$.

Since the fiducial metric (2.18) is flat in this region, there is no contribution from the curvature term in (2.17). For the kinetic term we have

$$\begin{aligned}
\tilde{z} &= \frac{1}{\delta^2 z} \approx \frac{1}{\delta^2 bt}, \quad \frac{d\tilde{z}}{dt} \approx -\frac{1}{\delta^2 b}\frac{1}{t^2}, \\
\partial_t\phi &\approx -\frac{2}{t}, \qquad \int d^2t\partial\phi\partial\phi \approx 32\pi\log\frac{\delta'}{|b|\delta}.
\end{aligned} \tag{2.36}$$

and we get

$$S_L^{(2)} = -\frac{1}{3}\log\frac{\delta'}{|b|\delta}. \tag{2.37}$$

(e) Moving further outwards in the t plane, we find a 'ring of curvature' at $|t| = 1/\delta'$ (cf. eq. (2.18)). At this ring we have

$$\phi \approx \log\frac{(\delta')^2}{\delta^2 b} + c.c., \qquad \int d^2tR\phi = 8\pi\phi, \tag{2.38}$$

which gives a contribution to S_L equal to

$$S_L^{(3)} = \frac{1}{3}\log\frac{(\delta')^2}{\delta^2|b|}. \tag{2.39}$$

The kinetic term in ϕ has no contribution at this ring. Further, the region $|t| > 1/\delta$ gives no contribution to S_L, since the curvature of the fiducial metric is zero, and the map gives $\phi = \text{constant}$.

2.6 The correlator in terms of the Liouville field

Let us collect all the above contributions together. Note that

$$S_L^{(2)} + S_L^{(3)} = \frac{1}{3}\log\frac{\delta'}{\delta}. \tag{2.40}$$

so that the variable b drops out of this combination.

Now let us go back to the expression (2.6) that we want to evaluate :

$$\langle \sigma_n^\epsilon(0)\sigma_n^\epsilon(a)\rangle_\delta = \frac{Z_{\epsilon,\delta}[\sigma_n(z_1),\sigma_n(z_2)]}{(Z_\delta)^N} = e^{S_L}\frac{Z^{(\hat{s})}}{(Z_\delta)^n}, \tag{2.41}$$

here we used equation (2.16). Taking into account the relations (2.19) and (2.20) we finally get:

$$\langle \sigma_n^\epsilon(0)\sigma_n^\epsilon(a)\rangle_\delta = e^{S_L}\left(\frac{\delta^n}{\delta'}\right)^{1/3} Q^{1-n}. \tag{2.42}$$

Substituting the expression for the Liouville action, S_L, we conclude that

$$\langle \sigma_n^\epsilon(0)\sigma_n^\epsilon(a)\rangle_\delta = e^{S_L^{(1)}} e^{S_L^{(2)}+S_L^{(3)}}\left(\frac{\delta^n}{\delta'}\right)^{1/3} Q^{1-n} = e^{S_L^{(1)}}\delta^{\frac{n-1}{3}}Q^{1-n}. \tag{2.43}$$

Thus we observe a cancellation of δ', which served only to choose a fiducial metric on Σ and thus should not appear in any final result. The only quantity that needs computation is $S_L^{(1)}$ using (2.33).

Let us mention that formula (2.41) has a simple extension to the case of a general correlation function:

$$\langle \sigma_{n_1}^\epsilon(z_1)\dots\sigma_{n_k}^\epsilon(z_k)\rangle_\delta = e^{S_L}\frac{Z^{(\hat{s})}}{(Z_\delta)^s}, \tag{2.44}$$

where $Z^{(\hat{s})}$ is a partition function of the covering Riemann surface Σ with the fiducial metric $d\hat{s}^2$ (Σ may have any genus), and s is a number of fields involved in nontrivial permutation ($s = n$ in the case of the two point function (2.41)). The partition function $Z^{(\hat{s})}$ may depend on the moduli of the surface Σ and its size (there are no moduli in the case of the sphere and the size is parameterized by δ').

3 The 2-point function

3.1 The calculation

Let us apply the above scheme to evaluate the 2-point function of twist operators. If one of the twist operators corresponds to the permutation

$$(1 \dots n), \tag{3.1}$$

then the other one should correspond to the permutation

$$(n \dots 1), \tag{3.2}$$

since otherwise the correlation function vanishes. Thus we can write

$$\langle \sigma_n(0)\sigma_n(a) \rangle \tag{3.3}$$

instead of $\langle \sigma_{(1\dots n)}(0)\sigma_{(n\dots 1)}(a) \rangle$ without causing confusion.

The generalization of the map (2.10) to the case of σ_n is

$$z = a\frac{t^n}{t^n - (t-1)^n}. \tag{3.4}$$

For this map we have

$$\begin{aligned} \phi &= \log|\frac{dz}{dt}|^2 = \log[an\frac{t^{n-1}(t-1)^{n-1}}{(t^n - (t-1)^n)^2}] + c.c., \\ \frac{d\phi}{dt} &= -\frac{(2t+n-1)(t-1)^n - (2t-n-1)t^n}{t(t-1)((t-1)^n - t^n)} \end{aligned} \tag{3.5}$$

This map has the branch points located at

$$t = 0 \to z = 0 \qquad \text{and} \qquad t = 1 \to z = a. \tag{3.6}$$

There are n images of the point $z = \infty$ in t plane:

$$t_k = \frac{1}{1-\alpha_k}, \qquad \alpha_k = e^{\frac{2\pi i k}{n}}, \qquad k = 0, 1, \dots, n-1. \tag{3.7}$$

We note that $\alpha_0 = 1$ gives $t = \infty$.

Let us compute the contribution (2.33) for the point $z = 0$. Near this point we have:

$$z \approx (-1)^{n+1} a t^n, \qquad |t| \approx \frac{|z|^{1/n}}{a^{1/n}}, \tag{3.8}$$
$$\phi \approx \log[ant^{n-1}], \qquad \partial_t \phi \approx \frac{n-1}{t}.$$

Then we get the contribution to the Liouville action (2.25):

$$S_L(t=0) = -\frac{n-1}{12}\left[\frac{1}{n}\log|a| + \log(n\epsilon^{\frac{n-1}{n}})\right]. \tag{3.9}$$

By a reflection symmetry $t \to 1-t$, $z \to a-z$, we get the same contribution from the other branch point:

$$S_L(t=1) = -\frac{n-1}{12}\left[\frac{1}{n}\log|a| + \log(n\epsilon^{\frac{n-1}{n}})\right]. \tag{3.10}$$

Now we look at the images of infinity. First we note that the integral over the boundary located near $t = \infty$ will give zero, since $d\phi/dt$ goes like $1/t^2$, the length of the circle goes like t and the value of ϕ is at best logarithmic in t. But we do get a contribution from the images of $z = \infty$ located at finite points in t plane. Note that

$$\text{if} \quad t = \frac{1}{1-\alpha_k} + x, \quad \text{then} \quad (t-1)^n - t^n \approx \frac{xn}{\alpha_k(1-\alpha_k)^{n-2}}. \tag{3.11}$$

This leads to

$$z \approx -\frac{a\alpha_k}{n(1-\alpha_k)^2}\,\frac{1}{x}, \qquad x \approx -\frac{a\alpha_k}{n(1-\alpha_k)^2}\,\frac{1}{z}, \tag{3.12}$$

$$\phi = \log\left[a^{-1} n(1-\alpha_k)^2 \alpha_k^{n-1} z^2\right], \qquad \partial_t\phi \approx -\frac{2}{t-(1-\alpha_k)^{-1}}. \tag{3.13}$$

The point $t = t_k$ we are considering gives following contribution to the Liouville action:

$$S_L(t=t_k) = \frac{1}{6}\log\left[a^{-1} n (1-\alpha_k)^2 \alpha_k^{n-1} \delta^{-2}\right]. \tag{3.14}$$

Thus the total contribution from the images of infinity is:

$$S_L(z=\infty) = \frac{1}{6}\sum_{k=1}^{n-1}\log\left[\frac{n(1-\alpha_k)^2\alpha_k^{n-1}}{a\delta^{-2}}\right] = -\frac{n-1}{6}\log[|a|\delta^2] + \frac{n+1}{6}\log[n]. \tag{3.15}$$

We have used the following properties of α_k:

$$\prod_{k=1}^{n-1} \alpha_k = 1; \qquad \prod_{k=1}^{n-1} (q - \alpha_k) = \frac{q^n - 1}{q - 1} \to n, \quad \text{if} \quad q \to 1, \tag{3.16}$$

which follow from the fact that $\{\alpha_k\}$ is the set of different solutions of the equation $\alpha^n - 1 = 0$ and $\alpha_0 = 1$.

Adding all the contributions together, we get an expression for the interesting part of the Liouville action:

$$S_L^{(1)} = -\frac{1}{6}\left[(n - \frac{1}{n}) \log |a| + \frac{(n-1)^2}{n} \log \epsilon + 2(n-1) \log \delta - 2 \log n\right]. \tag{3.17}$$

This leads to the final expression for the correlation function (see (2.43)):

$$\langle \sigma_n^\epsilon(0) \sigma_n^\epsilon(a) \rangle_\delta = e^{S_L^{(1)}} \delta^{\frac{n-1}{3}} Q^{1-n} = a^{-\frac{1}{6}(n - \frac{1}{n})} \, C_n \epsilon^{A_n} Q^{B_n}, \tag{3.18}$$

$$A_n = -\frac{(n-1)^2}{6n}, \quad B_n = 1 - n, \quad C_n = n^{1/3}. \tag{3.19}$$

Thus we read off the dimension Δ_n of σ_n

$$\Delta_n = \frac{1}{24}(n - \frac{1}{n}) \tag{3.20}$$

The other constants in (3.18) are to be absorbed into the normalization of σ_n. We will discuss this renormalization after computing the 3-point functions.

3.2 'Universality' of the 2-point function

The theory we have considered above is that of the orbifold M^N/S_N where the manifold M is just R, the real line. If M was R^d instead, we could treat the d different species of fields independently, and obtain

$$\Delta_n = \frac{c}{24}(n - \frac{1}{n}) \tag{3.21}$$

where $c = d$ is the central charge of the CFT for one copy of $M = R^d$. But we see that we would obtain the result (3.21) for the symmetric orbifold with *any* choice of M; we just use the value of c for the CFT on M. Around the insertion of the twist operator we permute the copies of M, but the

definition of the twist operator does not involve directly the structure of M itself. The Liouville action (2.17) determines the correlation function using only the value c of the CFT. Thus we recover the result (3.21) for any M.

This 'universality' of Δ_n is well known, and the value of Δ_n can be deduced from the following standard argument. Consider the CFT on a cylinder parameterized by $w = x + iy,\ 0 < y < 2\pi$. At $x \to -\infty$ let the state be the vacuum of the orbifold CFT M^N/S_N. Since there is no twist, each copy of M gives its own contribution to the vacuum energy, which thus equals $-\frac{c}{24}N$. Now insert the twist operator σ_n at $w = 0$, and look at the state for $x \to \infty$. The copies of M not involved in the twist contribute $-\frac{c}{24}$ each as before, but those that are twisted by σ_n turn into effectively one copy of M defined on a circle of length $2\pi n$. Thus the latter set contribute $-\frac{c}{24n}$ to the vacuum energy. The change in the energy between $x = \infty$ and $x = -\infty$ gives the dimension of σ_n (since the state at $x \to -\infty$ is the vacuum)

$$-\frac{c}{24n} - [-\frac{cn}{24}] = \frac{c}{24}(n - \frac{1}{n}) = \Delta_n \tag{3.22}$$

Thus while our calculation of the 2-point function has not taught us anything new, we have obtained a scheme that will yield the higher point functions for symmetric orbifolds using an extension of the same universal features that gave the value of Δ_n in the above argument.

4 The map for the 3-point function

4.1 Genus of the covering surface

Let us first discuss the nature of the covering surface Σ for the case where we have an arbitrary number of twist operators in the correlation function $\langle \sigma_{n_1}\sigma_{n_2}\ldots\sigma_{n_k}\rangle$. The CFT is still defined on the plane z, which we will for the moment regard as a sphere by including the point at infinity.

At the insertion of the operator $\sigma_{n_j}(z_j)$ the covering surface Σ has a branch point of order n_j, which means that n_j sheets of Σ meet at z_j. One says that the ramification order at z_j is $r_j = n_j - 1$. Suppose further that over a generic point z here are s sheets of the covering surface Σ. Then the genus g of Σ is given by the Riemann–Hurwitz formula:

$$g \ = \ \frac{1}{2}\sum_j r_j - s + 1 \tag{4.1}$$

Let us now consider the 3-point function. We require each twist operator to correspond to a single cycle of the permutation group, and regard the product of two cycles to represent the product of two different twist operators. Let the cycles have lengths n, m, q respectively. It is easy to see that we can obtain covering surfaces Σ of various genera. For example, if we have

$$\sigma_{12} \quad \sigma_{13} \quad \sigma_{123} \tag{4.2}$$

as the three permutations, then we have $r_1 = 1, r_2 = 1, r_3 = 2, s = 3$, and we get $g = 0$. On the other hand with

$$\sigma_{123} \quad \sigma_{123} \quad \sigma_{123} \tag{4.3}$$

we get $r_1 = r_2 = r_3 = 2, s = 3$ and we get $g = 1$. (This genus 1 surface is a singular limit of the torus, however.)

Let us concentrate on the case where we get $g = 0$. Without loss of generality we can take the first permutation σ_n to be the cycle

$$(1,\ 2,\ \dots\ k,\ k+1,\ \dots\ n) \tag{4.4}$$

The second permutation is restricted by the requirement that when composed with (4.4) it yields a single cycle (which would be the conjugate permutation of the third twist operator). In addition we must have a sufficiently small number of indices in the result of the first two permutations so that we do get $g = 0$. A little inspection shows that σ_m must have the form

$$(k,\ k-1,\ \dots\ 1,\ n+1,\ n+2,\ \dots\ n+m-k) \tag{4.5}$$

Thus the elements $1,\ 2,\ \dots\ k$ of the first permutation occur in the second permutation in the reverse order, and then we have a new set of elements $n+1,\ \dots\ n+m-k$. These two permutations compose to give the cycle $\sigma_m \sigma_n$ equal to

$$(\ k+1,\ k+2,\ \dots\ n,\ 1,\ n+1,\ n+2,\ \dots\ n+m-k) \tag{4.6}$$

Thus σ_q must be the inverse of the cycle (4.6), and we have

$$q = n + m - 2k + 1 \tag{4.7}$$

Note that the number of 'overlaps' (i.e., common indices) between σ_n and σ_m is k. Note that we must have $k \geq 1$ in order that the product $\sigma_m \sigma_n$ be

a single cycle rather than just a product of two cycles. Also note that if we have $q = n + m - 1$, then since $s \geq q$, (4.1) gives that Σ must have genus zero (this will be a 'single overlap' correlator).

Let Σ be the covering surface that corresponds to the insertions $\langle \sigma_n(z_1)\sigma_m(z_2)\sigma_q(z_3)\rangle$. Then the number of sheets of Σ over a generic point z is just the total number of indices used in the permutations

$$s = n + m - k = \frac{1}{2}(n + m + q - 1) \tag{4.8}$$

Thus the genus of Σ is

$$\frac{n-1}{2} + \frac{m-1}{2} + \frac{q-1}{2} - s + 1 = 0 \tag{4.9}$$

4.2 The map for the case $g = 0$

We are looking for a covering surface of the sphere that is ramified at three points on the sphere, with a finite order of ramification at each point. We look for the map from z to Σ as a ratio of two polynomials

$$z = \frac{f_1(t)}{f_2(t)}; \tag{4.10}$$

the existence of such a map will be evident from its explicit construction. By using the $SL(2, C)$ symmetry group of the z sphere, we will place the twist operators $\sigma_n, \sigma_m, \sigma_q$ at $z = 0, z = a, z = \infty$ respectively. We can assume without loss of generality that

$$n \leq q, \qquad m \leq q. \tag{4.11}$$

Note that we had placed a cutoff in the z plane to remove the region at infinity, and will not be immediately clear how to normalize a twist that occurs around the circle at infinity. We will discuss this issue of normalization later.

By making an $SL(2, C)$ transformation $t' = \frac{at+b}{ct+d}$ of the surface Σ, which we assume is parameterized by the coordinate t, we can take

$$z(t = 0) = 0, \qquad z(t = \infty) = \infty, \qquad z(t = 1) = a, \tag{4.12}$$

Note that this $SL(2, C)$ transformation maintains the form (4.10) of z to be a ratio of two polynomials, and we will use the symbols f_1, f_2 to denote the polynomials after the choice (4.12) has been made.

Since we need s values of t for a generic value of z, with s given by (4.8), the relation (4.10) should give a polynomial equation of order s for t. Thus the degrees d_1, d_2 of the polynomials f_1, f_2 should satisfy:

$$\max(d_1, d_2) = s = \frac{1}{2}(n + m + q - 1).$$

Since we have chosen $t = \infty$ for $z = \infty$, we get $d_1 > d_2$, and we have

$$d_1 = \frac{1}{2}(n + m + q - 1). \tag{4.13}$$

The requirement of the proper behavior at infinity ($z \sim t^q$) then gives:

$$d_2 = d_1 - q = \frac{1}{2}(n + m - q - 1). \tag{4.14}$$

Finally, the the number of indices common between the permutations $\sigma_n(0)$ and $\sigma_m(a)$ (the overlap) is

$$\frac{1}{2}(n + m - (q - 1)) = d_2 + 1. \tag{4.15}$$

Let us now look at the structure required of the map (4.10). For $z \to 0$ we need

$$z = t^n(C_0 + O(t)) \tag{4.16}$$

Similarly for $t \to 1$ we need

$$z = a + (t - 1)^m(C_1 + O(t - 1)) \tag{4.17}$$

Then we find

$$f_2^2 \frac{dz}{dt} = f_1' f_2 - f_2' f_1 = C t^{n-1}(1 - t)^{m-1}. \tag{4.18}$$

(C is a constant). The last step follows on noting that the expression $f_1' f_2 - f_2' f_1$ is a polynomial of degree $d_1 + d_2 - 1 = n + m - 2$, and the behavior of z near $z = 0, z = a$ already provides all the possible zeros of this polynomial $f_2^2 \frac{dz}{dt}$. The expression in (4.18) is just the Wronskian of f_1, f_2, and our knowledge of this Wronskian given an easy way to find these polynomials. We seek a second order linear differential equation whose solutions are the linear span $f = \alpha f_1 + \beta f_2$. Such an equation is found by observing that

$$\begin{vmatrix} f & f' & f'' \\ f_1 & f_1' & f_1'' \\ f_2 & f_2' & f_2'' \end{vmatrix} = 0 \tag{4.19}$$

so that we get the equation

$$Wf'' - W'f' + c(t)f = 0 \tag{4.20}$$

where

$$W = f_2 f_1' - f_1 f_2', \quad c(t) = f_2' f_1'' - f_1' f_2'' \tag{4.21}$$

Here W is given by (4.18). The coefficient $-W'$ of f' is

$$-W' = -Ct^{n-2}(t-1)^{m-2}[(n-1) - (n+m-2)t] \tag{4.22}$$

The coefficient $c(t)$ must be a polynomial of degree $n+m-4$ but in fact we can argue further that it must have the form

$$\gamma\, t^{n-2}(1-t)^{m-2}, \quad \gamma = \text{constant} \tag{4.23}$$

To see this look at the equation near $t=0$. Let $c(t) \sim \alpha t^k$ with $k < n-2$. Then the equation reads

$$t^{n-1-k} f'' - (n-1)t^{n-2-k} f' + \frac{\alpha}{C} f = 0 \tag{4.24}$$

Note that the two polynomials f_1, f_2 which solve the equation must not have a common root $t=0$, since we assume that (4.10) is already expressed in reduced form. Thus at least one of the solutions must go like $f \sim$ *constant* at $t=0$, which is in contradiction with (4.24) since the first two terms on the LHS vanish while the last does not ($a \neq 0$ by definition). Thus $c(t)$ has a zero of order at least $n-2$ at $t=0$, and by a similar argument, a zero of order at least $m-2$ at $t=1$. Thus the result (4.23) follows.

Dividing through by $Ct^{n-2}(1-t)^{m-2}$ we can write the equation (4.20) as

$$t(1-t)f'' + [-(n-1) + (n+m-2)t]f' + \tilde{\gamma} = 0 \tag{4.25}$$

Let us now look at $t \to \infty$, and let the solutions to the above equation go like t^p. Then we get

$$-p(p-1) + p(m+n-2) + \tilde{\gamma} = 0. \tag{4.26}$$

which has the solutions

$$p_{\pm} = \frac{1}{2}\left(m+n-1 \pm \sqrt{(m+n-1)^2 + 4\tilde{\gamma}}\right) \tag{4.27}$$

But since we have a twist operator of order q at infinity, we must have

$$p_+ - p_- = q, \tag{4.28}$$

This gives

$$\tilde{\gamma} = \frac{1}{4}(q - m - n + 1)(q + m + n - 1) = -d_1 d_2. \tag{4.29}$$

Thus we have found the equation which is satisfied by both f_1 and f_2:

$$t(1-t)y'' + (-n + 1 - (-d_1 - d_2 + 1)t)y' - d_1 d_2 y = 0, \tag{4.30}$$

which is the hypergeometric equation. Its general solution is given by

$$y = AF(-d_1, -d_2; -n + 1; t) + Bt^n F(-d_1 + n, -d_2 + n; n + 1; t). \tag{4.31}$$

The map we are looking for can be written as

$$z = a \frac{d_2! d_1!}{n!(d_1 - n)!} \frac{\Gamma(1-n)}{\Gamma(1 - n + d_2)} t^n \frac{F(-d_1 + n, -d_2 + n; n + 1; t)}{F(-d_1, -d_2; -n + 1; t)}, \tag{4.32}$$

where we have chosen the normalizations of f_1, f_2 such that the $t = 1$ maps to $z = a$. In our case d_1, d_2 and n are integers. Some of the individual terms in the above expression are undefined for integer d_1, d_2, n and a limit should be taken from non–integer values of n (while keeping d_1, d_2 fixed at their integer values). We can write the result in a well defined way by using Jacobi polynomials, which are a set of orthogonal polynomials defined through the hypergeometric function

$$\begin{aligned} P_n^{(\alpha,\beta)}(x) &\equiv \binom{n+\alpha}{n} F(-n, n + \alpha + \beta + 1; \alpha + 1; \frac{1-x}{2}) \\ &= \frac{1}{n!} \sum_{\nu=0}^{n} \binom{n}{\nu} (n + \alpha + \beta + 1) \dots (n + \alpha + \beta + \nu) \\ &\quad \cdot (\alpha + \nu + 1) \dots (\alpha + n) (\frac{x-1}{2})^\nu \end{aligned} \tag{4.33}$$

Then (4.32) becomes

$$z = at^n P_{d_1 - n}^{(n, -d_1 - d_2 + n - 1)}(1 - 2t) \left[P_{d_2}^{(-n, -d_1 - d_2 + n - 1)}(1 - 2t) \right]^{-1}. \tag{4.34}$$

We will have occasion to use the Wronskian of the polynomials later, and we define $\tilde{W}$ to be normalized as follows

$$\begin{aligned}\tilde{W}(t) &= \frac{d}{dt}\left[t^n P_{d_1-n}^{(n,-d_1-d_2+n-1)}(1-2t)\right] P_{d_2}^{(-n,-d_1-d_2+n-1)}(1-2t)\\ &- t^n P_{d_1-n}^{(n,-d_1-d_2+n-1)}(1-2t)\frac{d}{dt}P_{d_2}^{(-n,-d_1-d_2+n-1)}(1-2t)\\ &= \frac{nd_1!}{n!d_2!(d_1-n)!}\frac{\Gamma(d_2-n+1)}{\Gamma(1-n)}t^{n-1}(1-t)^{d_1+d_2-n},\end{aligned} \tag{4.35}$$

We will also have occasion to use the relation (4.32) containing hypergeometric functions, and we define

$$\begin{aligned}W(t) &= \left[t^n F(-d_1+n,-d_2+n;n+1;t)\right]' F(-d_1,-d_2;-n+1;t)\\ &- t^n F(-d_1+n,-d_2+n;n+1;t)F'(-d_1,-d_2;-n+1;t)\\ &= nt^{n-1}(1-t)^{d_1+d_2-n}.\end{aligned} \tag{4.36}$$

We will calculate the three point function using the map (4.32), (4.34) in the next section.

5 The Liouville action for the 3-point function

Let us evaluate the three point function

$$\langle\sigma_n(0)\sigma_m(a)\sigma_q(\infty)\rangle \tag{5.1}$$

using the map (4.32), (4.34).

Recall that we cut circles of radius ϵ in the z plane around the twist operators at $z=0$ and $z=a$ to regularize these twist operators. But unlike the case of the 2-point function discussed in section 3, now we have the twist operator σ_q inserted at infinity. This means that the fields X^I have boundary conditions around $z=\infty$ such that q of the X^I form a cycle under rotation around the circle $X^{i_1}\to X^{i_2}\to\dots X^{i_q}\to X^{i_1}$, while the remaining fields X^I are single valued around this circle. Note that if the covering surface Σ has s sheets over a generic z then there will be $s-q$ such single valued fields X^I.

The covering surface Σ will have punctures at $t=0$ and $t=1$ corresponding to $z=0$ and $z=a$ respectively. In addition it will have punctures corresponding to the 'puncture at infinity' in the z plane. These latter

punctures are of two kinds. The first kind of puncture in the t plane will correspond to the place where q sheets meet in the z plane - i.e., the lift of the point where the twist operator was inserted. But we will also have $s-q$ other punctures in the t plane that correspond to the cut at $|z| = 1/\delta$ for the X^I that are single valued around $z = \infty$. We will choose (when defining the 'regular region') a cutoff at value $|z| = 1/\tilde{\delta}$ for the first kind of puncture (i.e. the puncture arising from fields X^I that are twisted at $z = \infty$) and a value $|z| = 1/\delta$ for the second kind of puncture (i.e. punctures for fields X^I which are not twisted at infinity). We will see that both δ and $\tilde{\delta}$ cancel from all final results.

5.1 The contribution from $z = 0, t = 0$

Let us first consider the point $z = 0$ which gives $t = 0$. Near this point the map (4.32) gives:

$$z \approx a\frac{d_2!d_1!}{n!(d_1-n)!}\frac{\Gamma(1-n)}{\Gamma(1-n+d_2)}t^n, \qquad t \approx \left(\frac{zn!(d_1-n)!\Gamma(1-n+d_2)}{ad_2!d_1!\Gamma(1-n)}\right)^{1/n}. \tag{5.2}$$

Note that by using the relation

$$\Gamma(x)\Gamma(1-x) = \frac{\pi}{\sin(\pi x)} \tag{5.3}$$

we can write

$$\frac{\Gamma(1-n)}{\Gamma(1-n+d_2)} = \frac{\Gamma(n-d_2)}{\Gamma(n)}\frac{\sin(\pi(n-d_2))}{\sin(\pi n)} = \frac{(n-1)!}{(n-d_2-1)!}(-1)^{d_2}, \tag{5.4}$$

so that the Γ functions in the above expressions are in reality well defined.

The Liouville field and its derivative are given by:

$$\phi \approx \log\left(\frac{nad_2!d_1!}{n!(d_1-n)!}\frac{(n-d_2-1)!}{(n-1)!}t^{n-1}\right) + c.c., \qquad \partial_t\phi \approx \frac{n-1}{t}, \tag{5.5}$$

where we have dropped the factor $(-1)^{d_2}$ in (5.4) since ϕ is the real part of the logarithm. Substituting these values into the expression for the Liouville action:

$$S_L = \frac{i}{96\pi}\int dt\phi\partial_t\phi, \tag{5.6}$$

we get a contribution from the point $t=0$:

$$S_L(t=0) = -\frac{n-1}{12}\log\left(n\epsilon^{\frac{n-1}{n}}\right) - \frac{n-1}{12n}\log\left(a\frac{d_2!d_1!}{n!(d_1-n)!}\frac{(n-d_2-1)!}{(n-1)!}\right), \tag{5.7}$$

where we note that the integration in (5.6) is performed along the circle

$$|t| = \left(\frac{\epsilon n!(d_1-n)!(n-1)!}{ad_2!d_1!(n-d_2-1)!}\right)^{1/n}. \tag{5.8}$$

A simplification analogous to (5.4) will occur in many relations below, but for simplicity we leave the Γ functions in the form where they have negative arguments; we replace them with factorials of positive numbers only in the final expressions.

5.2 The contribution from $z=a, t=1$

Let us look at the point $t=1$. Using the expression for Wronskian (4.36), we find the derivative of the map (4.32):

$$\frac{dz}{dt} = a\frac{d_2!d_1!}{n!(d_1-n)!}\frac{\Gamma(1-n)}{\Gamma(1-n+d_2)}\frac{nt^{n-1}(1-t)^{d_1+d_2-n}}{\left[F(-d_1,-d_2;-n+1;t)\right]^2}, \tag{5.9}$$

which can be combined with known property of hypergeometric function:

$$F(a,b;c;1) = \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)} \tag{5.10}$$

to give the result:

$$z \approx a - \beta a(1-t)^{d_1+d_2-n+1}, \tag{5.11}$$

$$\beta = \frac{n}{d_1+d_2-n+1}\frac{d_1!d_2!(d_1-n)!}{n!\left[(d_1+d_2-n)!\right]^2}\frac{\Gamma(1-n+d_2)}{\Gamma(1-n)}. \tag{5.12}$$

Our usual analysis gives:

$$\begin{aligned} z &\approx a - \beta a(1-t)^{d_1+d_2-n+1}, \qquad 1-t \approx \left(-\frac{z-a}{a\beta}\right)^{\frac{1}{d_1+d_2-n+1}}, \\ \phi &\approx \log\left(a\beta(d_1+d_2-n+1)(1-t)^{d_1+d_2-n}\right) + c.c., \quad \partial_t\phi \approx -\frac{d_1+d_2-n}{1-t}, \end{aligned}$$

$$\begin{aligned} S_L(t=1) &= -\frac{d_1+d_2-n}{12}\log(d_1+d_2-n+1) \qquad (5.13) \\ &- \frac{(d_1+d_2-n)^2}{12(d_1+d_2-n+1)}\log\epsilon - \frac{(d_1+d_2-n)}{12(d_1+d_2-n+1)}\log(a|\beta|). \end{aligned}$$

Note that $d_1+d_2-n+1=m$, so that we can rewrite the contribution from $t=1$ in a way which makes it look more symmetrical with the contribution from $t=0$. But we will defer all such simplifications to the final expressions for the fusion coefficients.

5.3 The contribution from $z=\infty$

To analyze the contribution from the point $t=\infty$ it is convenient to look at the map written in terms of Jacobi polynomials (4.34). Then one can use the Rodrigues' formula to represent the Jacobi polynomials in the form:

$$P_k^{(\alpha\beta)}(x) = 2^{-k}\sum_{j=0}^{k}\begin{pmatrix} k+\alpha \\ j \end{pmatrix}\begin{pmatrix} k+\beta \\ k-j \end{pmatrix}(x-1)^{k-j}(x+1)^j. \qquad (5.14)$$

The limit $x\to\infty$ gives:

$$P_k^{(\alpha\beta)}(x) \to x^k 2^{-k}\sum_{j=0}^{k}\begin{pmatrix} k+\alpha \\ j \end{pmatrix}\begin{pmatrix} k+\beta \\ k-j \end{pmatrix} = x^k 2^{-k}\begin{pmatrix} 2k+\alpha+\beta \\ k \end{pmatrix}. \qquad (5.15)$$

Substitution of this limit into the expression (4.34) gives the behavior near $t=\infty$:

$$\begin{aligned} z &\approx a\gamma(-1)^{d_1-d_2-n}t^{d_1-d_2}, \qquad t\approx\left((-1)^{d_1-d_2-n}\frac{z}{a\gamma}\right)^{\frac{1}{d_1-d_2}} \\ \gamma &= \frac{d_2!(d_1-d_2-1)!}{(d_1-n)!(n-d_2-1)!}\,\frac{\Gamma(-d_1)}{\Gamma(d_2-d_1)}, \qquad (5.16) \\ \phi &\approx \log\left(a\gamma(d_1-d_2)t^{d_1-d_2-1}\right)+c.c., \qquad \partial_t\phi\approx\frac{d_1-d_2-1}{t}. \end{aligned}$$

Consider first the point $z=\infty, t=\infty$. Recall that we have taken the 'regular region' on Σ to be bounded by the image of $1/\tilde{\delta}$ (rather than $1/\delta$) when a twist operator is inserted. The contour around the puncture at infinity in the t plane should be taken to go clockwise rather than anti-clockwise, so that it looks like a normal anti-clockwise contour in the local

coordinate $t' = 1/t$ around the puncture. Thus to compute the contribution from this puncture we should follow our usual procedure but reverse the overall sign. The result reads:

$$\begin{aligned} S_L(t=\infty) &= (-1)\left[-\frac{d_1-d_2-1}{12}\log(d_1-d_2)\right. \\ &+ \left.\frac{(d_1-d_2-1)^2}{12(d_1-d_2)}\log\tilde{\delta} - \frac{d_1-d_2-1}{12(d_1-d_2)}\log(a|\gamma|)\right]. \end{aligned} \tag{5.17}$$

Finally let us analyze the images of $z=\infty$ that give finite values t_i of t. At each of these points the map $t \to z$ is one–to–one, in contrast to the above case $z=\infty, t=\infty$ where q values of t correspond to each value of z in a neighborhood of the puncture. Further, there is no sign reversal for the contour of integration around these punctures when we use the coordinate t to describe the contour.

Looking at the structure of the map (4.34) one can easily identify the locations of the t_i: they coincide with zeroes of the denominator. So to evaluate the contribution to the Liouville action from the t_i we will need some information about zeroes of Jacobi polynomials. Using the fact that Jacobi polynomials have only simple zeroes we can expand the map (4.34) around any of the t_i:

$$z \approx a t_i^n \frac{P_{d_1-n}^{(n,-d_1-d_2+n-1)}(1-2t_i)}{P'^{(-n,-d_1-d_2+n-1)}_{d_2}(1-2t_i)}\,\frac{1}{t-t_i} \equiv \frac{a\xi_i}{t-t_i}, \tag{5.18}$$

$$\xi_i = t_i^n \frac{P_{d_1-n}^{(n,-d_1-d_2+n-1)}(1-2t_i)}{\frac{d}{dt}P_{d_2}^{(-n,-d_1-d_2+n-1)}(1-2t_i)}. \tag{5.19}$$

Then everything can be evaluated in terms of ξ_i:

$$\begin{aligned} t-t_i &\approx \frac{a\xi_i}{z}, \qquad \phi \approx \log\left(\frac{-a\xi_i}{(t-t_i)^2}\right) + c.c., \qquad \partial_t\phi \approx -\frac{2}{t-t_i} \\ S_L(t=t_i) &= -\frac{1}{6}\log(\delta^2 a\xi_i). \end{aligned} \tag{5.20}$$

Collecting the contributions from all the t_i we get:

$$S_L(\text{all } t_i) = -\frac{d_2}{6}\log(\delta^2 a) - \frac{1}{6}\sum_{i=1}^{d_2}\log(\xi_i), \tag{5.21}$$

and we only need to evaluate the product of ξ_i. Note that the regularization parameter δ we use here has the same meaning as one considered in section 3.

This product can be written in terms of the Wronskian (4.35) and the discriminant of Jacobi polynomials. To see this we first rewrite (5.19) in terms of zeroes of Jacobi polynomials. If $z = P/Q$, then the Wronskian (4.35) is $\tilde{W} = P'Q - PQ' = -PQ'$ at a zero of Q. Writing any of the ξ_i as $\xi = P/Q' = PQ'/Q'^2$ we find

$$\xi_i = -\tilde{W}(t_i)\left[-2a_0 \prod_{j\neq i}(x_i - x_j)\right]^{-2}, \tag{5.22}$$

where a_0 is the coefficient in front of the highest power in the polynomial $P_{d_2}^{(-n,-d_1-d_2+n-1)}$; it can be evaluated using (5.15). The x_i are the zeros of the polynomial $Q(x)$ in the denominator.

Applying the general definition of the discriminant to Jacobi polynomials

$$D_{d_2}^{(-n,-d_1-d_2+n-1)} \equiv a_0^{2d_2-2}\prod_{i<j}(x_i - x_j)^2, \tag{5.23}$$

we get

$$\prod_{i=1}^{d_2}\xi_i = (-1)^{d_2}2^{-2d_2}a_0^{2(d_2-2)}\left[D_{d_2}^{(-n,-d_1-d_2+n-1)}\right]^{-2}\prod_{i=1}^{d_2}\tilde{W}(t_i). \tag{5.24}$$

The discriminant of Jacobi polynomials can be evaluated [17]:

$$\begin{aligned} \mathcal{D} &\equiv D_{d_2}^{(-n,-d_1-d_2+n-1)} = 2^{-d_2(d_2-1)} \\ &\times \prod_{j=1}^{d_2} j^{j+2-2d_2}(j-n)^{j-1}(j-d_1-d_2+n-1)^{j-1}(j-d_1-1)^{d_2-j}. \end{aligned} \tag{5.25}$$

To evaluate the right hand side of (5.24) we only need the expressions for

$$\prod_{i=1}^{d_2} t_i \qquad \text{and} \qquad \prod_{i=1}^{d_2}(1-t_i). \tag{5.26}$$

Let us consider the general Jacobi polynomial:

$$P_k^{(\alpha\beta)}(1-2t) = (-2)^k a_0 t^k + \ldots + a_{k+1} = (-2)^k b_0 (t-1)^k + \ldots + b_{k+1}, \tag{5.27}$$

Obviously $b_0 = a_0$. By taking the limits $t \to \infty$, $t \to 0$ and $t \to 1$ in the above expression we find:

$$\prod_{i=1}^{d_2} t_i = \frac{a_{k+1}}{2^k a_0} = \frac{\Gamma(k+\alpha+1)\Gamma(k+\alpha+\beta+1)}{\Gamma(\alpha+1)\Gamma(2k+\alpha+\beta+1)}, \tag{5.28}$$

$$\prod_{i=1}^{d_2}(1-t_i) = \frac{b_{k+1}}{2^k b_0} = \frac{\Gamma(k+\beta+1)\Gamma(k+\alpha+\beta+1)}{\Gamma(\beta+1)\Gamma(2k+\alpha+\beta+1)}. \tag{5.29}$$

Collecting all contributions together, we get

$$\begin{aligned}
\log\prod_{i=1}^{d_2}\xi_i &= -2d_2(d_2-1)\log 2 + d_2\log n - 2\log\mathcal{D} - (3d_2-4)\log d_2! \\
&+ d_2\log\left[\frac{d_1!}{n!(d_1-n)!}\right] + (n+d_2-1)\log\frac{(n-1)!}{(n-d_2-1)!} \\
&+ (d_1-d_2+3)\log\frac{(d_1-d_2)!}{d_1!} + (d_1+d_2-n)\log\frac{(d_1+d_2-n)!}{(d_1-n)!}.
\end{aligned} \tag{5.30}$$

5.4 The total Liouville action.

Collecting the contributions from the different branching points we obtain the final expression for the Liouville action

$$\begin{aligned}
S_L^{(1)} &= -\left(\frac{(n-1)^2}{12n} + \frac{(d_1+d_2-n)^2}{12(d_1+d_2-n+1)}\right)\log\epsilon - \frac{(d_1-d_2-1)^2}{12(d_1-d_2)}\log\tilde{\delta} \\
&- \frac{d_2}{3}\log\delta - \left(\frac{n-1}{12n} + \frac{d_1+d_2-n}{12(d_1+d_2-n+1)} - \frac{d_1-d_2-1}{12(d_1-d_2)} + \frac{d_2}{6}\right)\log a \\
&- \frac{n-1}{12}\log n - \frac{d_1+d_2-n}{12}\log(d_1+d_2-n+1) \\
&+ \frac{d_1-d_2-1}{12}\log(d_1-d_2) - \frac{n-1}{12n}\log\left(\frac{d_1!d_2!}{n!(d_1-n)!}\,\frac{\Gamma(1-n)}{\Gamma(1-n+d_2)}\right) \\
&- \frac{d_1+d_2-n}{12(d_1+d_2-n+1)}\log|\beta| + \frac{d_1-d_2-1}{12(d_1-d_2)}\log|\gamma| - \frac{1}{6}\log\left(\prod_{i=1}^{d_2}\xi_i\right).
\end{aligned} \tag{5.31}$$

The values of β and γ are given by (5.12) and (5.16), and the last term is given through (5.30). According to (2.44), the three point function is given

by:

$$\langle\sigma_n^\epsilon(0)\sigma_m^\epsilon(a)\sigma_q^{\tilde{\delta}}(\infty)\rangle_\delta = e^{S_L^{(1)}} e^{S_L^{(2)}+S_L^{(3)}} \frac{Z^{(\hat{s})}}{(Z_\delta)^s}, \tag{5.32}$$

where s is number of fields involved in permutation; it is defined by (4.8). Note that we have not determined the values of $S_L^{(2)}$ and $S_L^{(3)}$ for the case under consideration, as we will see these quantities will cancel in the final answer.

6 Normalizing the twist operators

This Liouville action (5.31) yields the correlation function for twist operators with the regularization parameters $\epsilon, \delta, \tilde{\delta}$. We immediately see that the power of a in the correlator is

$$\begin{aligned} a: \qquad & -\left(\frac{n-1}{12n} + \frac{d_1+d_2-n}{12(d_1+d_2-n+1)} - \frac{d_1-d_2-1}{12(d_1-d_2)} + \frac{d_2}{6}\right) \\ & + \frac{1}{2}\left(M_a^n + M_a^m - M_a^q\right) = -\frac{1}{6}\left(n - \frac{1}{n} + m - \frac{1}{m} - q + \frac{1}{q}\right), \end{aligned} \tag{6.1}$$

which agrees with the expected a dependence of the 3-point function

$$\langle\sigma_n^\epsilon(0)\sigma_m^\epsilon(a)\sigma_q^{\tilde{\delta}}(\infty)\rangle \;\sim\; |a|^{-2(\Delta_m+\Delta_n-\Delta_q)}. \tag{6.2}$$

To obtain the final correlation functions and fusion coefficients we have two sources of renormalization coefficients that need to be considered:

(a) We have to normalize the operators σ_n^ϵ such that their 2-point functions are set to unity at unit z separation; at this point we should find that the parameters $\epsilon, \delta, \tilde{\delta}$ disappear from the 3-point (and higher point) functions as well. After we normalize the twist operators σ_n^ϵ in this way we will call them σ_n.

(b) The CFT had N fields X^I, though only n of them are affected by the twist operator σ_n^ϵ. However at the end of the calculation of any correlation function of the operators $\sigma_{n_i}^\epsilon$ we must sum over all the possible ways that the n_i fields that are twisted can be chosen from the total set of N fields. Thus we will have to define operators O_n that are sums over conjugacy classes of the permutation group, and these operators O_n are the only ones that will finally be well defined operators in the CFT [10]. The correctly normalized

O_n will thus have combinatoric factors multiplying the normalized operators σ_n.

We choose to arrive at the final normalized operators O_n in these two steps since the calculations involved in steps (a) and (b) are quite different; further when $n_i << N$ the factor coming from (b) is just a power of N which is easily found.

6.1 Normalizing the σ_n^ϵ

Let us define the normalized twist operators σ_n by requiring

$$\langle\sigma_n(0)\sigma_n(a)\rangle \;=\; \frac{1}{|a|^{4\Delta_n}}. \tag{6.3}$$

From (3.18) we see that

$$\sigma_n = D_n\sigma_n^\epsilon, \qquad D_n = [C_n\epsilon^{A_n}Q^{B_n}]^{-1/2}. \tag{6.4}$$

Let the Operator Product Expansion (OPE) have the form

$$\sigma_m(a)\sigma_n(0) \sim \frac{|C^\sigma_{nmq}|^2}{|a|^{2(\Delta_n+\Delta_m-\Delta_q)}}\;\sigma_q(0)+\ldots, \tag{6.5}$$

where we have written the OPE for holomorphic and anti-holomorphic blocks combined, and we have put a superscript σ on the fusion coefficients C^σ_{nmq} to remind ourselves that these are not the final fusion coefficients of the physical operators O_n. With the normalization (6.3) we will get

$$\langle\sigma_n(z_1)\sigma_m(z_2)\sigma_q(z_3)\rangle = \frac{|C^\sigma_{nmq}|^2}{|z_1-z_2|^{2(\Delta_n+\Delta_m-\Delta_q)}|z_2-z_3|^{2(\Delta_m+\Delta_q-\Delta_n)}|z_3-z_1|^{2(\Delta_q+\Delta_n-\Delta_m)}} \tag{6.6}$$

We have computed the 3-point functions and should thus be able to get the fusion coefficients C^σ_{nmq} from (6.6). However while two of our twist operators were inserted at finite points in the z plane, the last one was inserted at infinity. Putting one of the points at infinity simplified the calculation, but it also creates the following problem: unlike the twist operators at $z=0, z=a$ which are normalized through (6.4) it is not clear what is the normalization of the twist operator that is inserted at infinity. (We could think of this operator as inserted at a puncture on the sphere at infinity, and therefore

no different from the other insertions, but we have chosen the flat metric on the z plane and thus made infinity a special region carrying curvature.) To get around this problem we adopt the following scheme. If we have the OPE (6.5) then we will get

$$\frac{\langle\sigma_n(0)\sigma_m(a)\sigma_q^{\tilde\delta}(\infty)\rangle}{\langle\sigma_q(0)\sigma_q^{\tilde\delta}(\infty)\rangle}=\frac{|C^\sigma_{nmq}|^2}{|a|^{2(\Delta_n+\Delta_m-\Delta_q)}} \tag{6.7}$$

and we will thus not need to know the normalization of the operator at infinity. $\langle\sigma_n(0)\sigma_m(a)\sigma_q^{\tilde\delta}(\infty)\rangle$ can be found from our 3-point function calculation together with the normalization factors for σ_n, σ_m from (6.3). To compute the denominator we must find the 2-point function with one operator at infinity. (We had earlier computed the 2-point function with both operators in the finite z plane since if we put one operator at infinity then we loose any position dependence in the correlator and cannot extract the scaling dimensions.)

To evaluate the two point function

$$\langle\sigma_n(0)\sigma_n(\infty)\rangle \tag{6.8}$$

we consider the map:

$$z = bt^n. \tag{6.9}$$

This map has order n ramification points at $t = 0$ and $t = \infty$ and the usual calculations give:

$$\phi = \log[nbt^{n-1}] + c.c., \qquad \partial_t\phi = \frac{n-1}{t},$$

$$S_L(t=0) = -\frac{n-1}{12}\log[nb^{1/n}\epsilon^{(n-1)/n}], \tag{6.10}$$

$$S_L(t=\infty) = -(-1)\frac{n-1}{12}\log[nb^{1/n}\tilde\delta^{(1-n)/n}]. \tag{6.11}$$

As before we cut a hole of size ϵ around the origin and put the twist at infinity on a boundary at $|z| = 1/\tilde\delta$. We also have an extra negative sign for the cut at infinity since the contour that goes anti-clockwise in the local coordinate at $1/t$ near $t = \infty$ goes clockwise in the coordinate t. Collecting both contributions we get:

$$\langle\sigma_n^\epsilon(0)\sigma_n^{\tilde\delta}(\infty)\rangle = \epsilon^{F_n}\tilde\delta^{F_n}e^{S_L^{(2)}+S_L^{(3)}}\frac{Z^{(\hat s)}}{(Z_\delta)^n}, \tag{6.12}$$

$$F_n = -\frac{(n-1)^2}{12n}. \tag{6.13}$$

The correlator is b–independent as expected. The values of $S_L^{(2)}$, $S_L^{(3)}$ and partition function $Z^{(\hat{s})}$ depend on $\delta, \delta', \tilde{\delta}$, but these expressions are are the same as in (5.32) and so they cancel in the final answer.

The $C^\sigma_{n,m,q}$ are then given by

$$
\begin{aligned}
|C^\sigma_{n,m,q}|^2 &= \frac{\langle \sigma_n(0)\sigma_m(a)\sigma_q^{\tilde{\delta}}(\infty)\rangle}{\langle \sigma_q(0)\sigma_q^{\tilde{\delta}}(\infty)\rangle} \\
&= \frac{\langle \sigma^\epsilon_n(0)\sigma^\epsilon_m(a)\sigma_q^{\tilde{\delta}}(\infty)\rangle}{\langle \sigma^\epsilon_q(0)\sigma_q^{\tilde{\delta}}(\infty)\rangle}\sqrt{\frac{\langle \sigma^\epsilon_q(0)\sigma^\epsilon_q(1)\rangle}{\langle \sigma^\epsilon_n(0)\sigma^\epsilon_n(1)\rangle\langle \sigma^\epsilon_m(0)\sigma^\epsilon_m(1)\rangle}}\,.
\end{aligned} \tag{6.14}
$$

As a consistency check of our procedure we look at powers of various regularization parameters: they should cancel in any physical quantity. For $|C^\sigma_{n,m,q}|^2$ we have the following powers for $\epsilon, \delta, \tilde{\delta}$, Q:

$$\epsilon: \quad -\left(\frac{(n-1)^2}{12n}+\frac{(m-1)^2}{12m}\right)-F_q-\frac{1}{2}\left(A_n+A_m-A_q\right)=0, \tag{6.15}$$

$$\tilde{\delta}: \quad -\frac{(q-1)^2}{12q}-F_q=0, \tag{6.16}$$

$$\delta: \quad -\frac{d_2}{3}-s+q=0, \tag{6.17}$$

$$Q: \quad 1-s-\frac{1}{2}\left(B_n+B_m+B_q\right)=0. \tag{6.18}$$

We used the expressions (4.13) and (4.14) for d_1 and d_2, the values of A_n, B_n, F_n from (3.19) and (6.13) and the genus relation (4.9).

We finally get (for the contribution from Σ of genus zero) the logarithm of the fusion coefficient

$$
\begin{aligned}
\log|C^\sigma_{n,m,q}|^2 &= \frac{1}{6}\log\left(\frac{q}{mn}\right)-\frac{n-1}{12}\log n-\frac{m-1}{12}\log m+\frac{q-1}{12}\log(q) \\
&- \frac{n-1}{12n}\log\left(\frac{d_1!d_2!}{n!(n-1)!}\,\frac{(d_1-m)!}{(d_1-n)!}\right) \\
&- \frac{m-1}{12m}\log\left(\frac{d_1!d_2!}{m!(m-1)!}\,\frac{(d_1-n)!}{(d_1-m)!}\right) \\
&+ \frac{q-1}{12q}\log\left(\frac{(q-1)!d_2!}{(d_1-n)!(d_1-m)!}\,\frac{(d_1-d_2)!)}{d_1!}\right)-\frac{1}{6}\log\left(\prod_{i=1}^{d_2}\xi_i\right).
\end{aligned} \tag{6.19}
$$

The expression for the product of ξ_i is given by (5.30).

The coefficients $C^{\sigma}_{n,m,q}$ must be symmetric in the indices m, n, q. We have written (6.20) in such a way that all the terms except the last one show a manifest symmetry between m and n. It can be shown without much difficulty that the last term (given through (5.30) and (5.25)) is also symmetric in m and n. In particular d_1 and d_2 are symmetric in m and n, and $-d_1-d_2+n-1=-m$, so that the Jacobi polynomial whose discriminant is calculated in (5.25) is $P^{-n,-m}_{d_2}$.

On the other hand it is not at all obvious that the expression (5.30) is symmetric under the interchange of q with either n or m. Note that d_2+1 is the number of elements that overlap between the permutations σ_n and σ_m, and the product in (5.25) runs over the range $j=1\dots d_2$. This number d_2+1 is in general different from the number of overlapping elements between the permutations σ_q and σ_n or between σ_q and σ_m, and thus there is no simple way to write (5.30) in form that makes its total symmetry manifest. Nevertheless, this expression is indeed symmetric in all three arguments n, m, q, as can be checked by evaluating the expression through a symbolic manipulation program. Verifying this symmetry provides a useful check of all our calculations for the 3-point function.

6.2 Two special cases

Due to the structure of the discriminant (5.25), the general expression for the fusion coefficient looks complicated for an arbitrary value of d_2. However there are two important cases where significant simplifications occur. These are the cases of one and two overlaps (we recall that the number of common indices in σ_n and σ_m is d_2+1). One can see that for $d_2=0$ and $d_2=1$ the discriminant $\mathcal{D}=1$. Let us analyze both these cases.

For one overlap we have:

$$d_2 = 0, \qquad d_1 = q = m+n-1, \tag{6.20}$$

and the logarithm of fusion coefficient is given by:

$$\begin{aligned}\log|C^{\sigma}_{n,m,m+n-1}|^2 &= -\frac{1}{12}\left(n+\frac{1}{n}\right)\log n - \frac{1}{12}\left(m+\frac{1}{m}\right)\log m \\ &+\frac{1}{12}\left(q+\frac{1}{n}+\frac{1}{m}-1\right)\log q - \frac{1}{12}\left(1+\frac{1}{q}-\frac{1}{n}-\frac{1}{m}\right)\log\left(\frac{(q-1)!}{(m-1)!(n-1)!}\right).\end{aligned} \tag{6.21}$$

In particular we get:

$$|C^{\sigma}_{223}|^2 = 2^{-\frac{4}{9}} 3^{\frac{1}{4}}. \tag{6.22}$$

The case of two overlaps corresponds to

$$d_2 = 1, \qquad q = m + n - 3, \qquad d_1 = q + 1, \tag{6.23}$$

and the result reads:

$$\log |C^{\sigma}_{n,m,m+n-3}|^2 = \frac{1}{12}\left(\frac{1}{n}+\frac{1}{m}-\frac{1}{m+n-3}-3\right)\log\left(\frac{(m+n-3)!}{(m-1)!(n-1)!}\right)$$
$$-\frac{n^2+1}{12n}\log n - \frac{m^2+1}{12m}\log m + \frac{1}{12}\left(2+\frac{(m+n-4)^2}{m+n-3}\right)\log(m+n-3)+$$
$$\frac{1}{12}\left(\frac{n-m}{mn}-2n+\frac{m+n-4}{m+n-3}\right)\log(n-1) + \frac{1}{12}\left(\frac{m-n}{mn}-2m+\frac{m+n-4}{m+n-3}\right)\log(m-1)$$
$$+\frac{1}{12}\left(2(m+n)-5+\frac{1}{n}+\frac{1}{m}+\frac{1}{m+n-3}\right)\log(m+n-2). \tag{6.24}$$

In particular for $m = 3$ we get:

$$\begin{aligned} \log |C^{\sigma}_{n3n}|^2 &= -\frac{2}{9}\log n - \frac{1}{6}\left(n+\frac{1}{n}-\frac{2}{3}\right)\log(n-1) \\ &+ \frac{1}{6}\left(n+\frac{1}{n}+\frac{2}{3}\right)\log(n+1) - \frac{2}{9}\log 2 - \frac{5}{18}\log 3. \end{aligned} \tag{6.25}$$

From this expression we can extract the value of C^{σ}_{232} and check that it equals the value of C^{σ}_{223} given by (6.22).

6.3 Combinatoric factors and large N limit.

The twist operators we have considered so far do not represent proper fields in the conformal field theory. In the orbifold CFT there is one twist field for each conjugacy class of the permutation group, not for each element of the group [10]. The true CFT operators that represent the twist fields can be constructed by summing over the group orbit:

$$O_n = \frac{\lambda_n}{N!}\sum_{h\in G}\sigma_{h(1\dots n)h^{-1}} \ . \tag{6.26}$$

Here G is the permutation group S_N and the normalization constant λ_n will be determined below.

Using normalization condition for the σ operators:

$$\langle \sigma_n(0)\sigma_n(1)\rangle = 1 \tag{6.27}$$

we find:

$$\langle O_n(0)O_n(1)\rangle = \frac{\lambda_n^2}{N!}\langle \sigma_{(1\ldots n)} \sum \sigma_{h(1\ldots n)h^{-1}}\rangle = \lambda_n^2 n \frac{(N-n)!}{N!}\langle \sigma_n(0)\sigma_n(1)\rangle. \tag{6.28}$$

Requiring the normalisation $\langle O_n(0)O_n(1)\rangle = 1$ we find the value of λ_n:

$$\lambda_n = \left[\frac{n(N-n)!}{N!}\right]^{-1/2}. \tag{6.29}$$

Let us now look at the three point function. First we consider the combinatorics for the $g = 0$ cases that we worked with above; the permutation structure was described in (4.4), (4.5), (4.6). Simple combinatorics yields

$$\begin{aligned}\langle O_n(0)O_m(1)O_q(z)\rangle &= (\frac{1}{N!})^3\lambda_n\lambda_m\lambda_q \\ \times nmq\frac{N!}{(N-s)!}&(N-n)!(N-m)!(N-q)!\langle \sigma_n(0)\sigma_m(1)\sigma_q(z)\rangle.\end{aligned} \tag{6.30}$$

One way of getting this expression is to note that s different indices are involved in the permutation, and we can select these indices, in the order in which they appear when the permutations are written out, in $N!/(N-s)!$ ways. Having obtained the indices for any given permutation, we ask how many elements out of the sum over group elements yields this set of indices in the permutation; the answer for σ_n for example is $(N-n)!$, since only the permutations of the remaining $N-n$ elements leave the indices in σ_n untouched. Finally, we note that any permutation σ_k can be written in k equivalent ways since we can begin the set of indices with any index that we choose from the set; this leads to the factors nmq.

Substituting the values of λ_i we get the final result:

$$\langle O_n(0)O_m(1)O_q(z)\rangle = \frac{\sqrt{mnq(N-n)!(N-m)!(N-q)!}}{(N-s)!\sqrt{N!}}\langle \sigma_n(0)\sigma_m(1)\sigma_q(z)\rangle. \tag{6.31}$$

with $s = \frac{1}{2}(n+m+q-1)$.

Now we analyse the behavior of the combinatoric factors for arbitrary genus g but in the limit where N is taken to be large while the orders of twist operators (m, n and q) as well as the parameter g are kept fixed. There are s different fields X^i involved in the 3-point function, and these fields can be selected in $\sim N^s$ ways. Similarily the 2-point function of σ_n will go as N^n since n different fields are to be selected. Thus the 3-point function of normalised twist operators will behave as

$$N^{s-\frac{n+m+q}{2}} = N^{-(g+\frac{1}{2})} \tag{6.32}$$

(which can also be obtained from (6.31)).

Thus in the large N limit the contributions from surfaces with high genus will be suppressed, and the leading order the answer can be obtained by considering only contributions from the sphere ($g = 0$). This is presicely the case that we have analysed in detail, and knowing the amplitude $\langle\sigma_n(0)\sigma_m(1)\sigma_q(z)\rangle$ one can easily extract the leading order of the CFT correlation function:

$$\langle O_n(0)O_m(1)O_q(z)\rangle = \sqrt{\frac{1}{N}}\sqrt{mnq}\langle\sigma_n(0)\sigma_m(1)\sigma_q(z)\rangle_{sphere} + O\left(\frac{1}{N^{3/2}}\right). \tag{6.33}$$

7 Four Point Function.

In this section we compute specific examples of 4-point functions, without attempting to analyze the most general case. The computations illustrate interesting features which arise in our approach for four and higher point functions. In particular we will also need to compute a genus one correlation function. We will also be able to verify specific examples of the fusion coefficients computed in the last section as they will be recovered through factorization of the 4-point functions.

7.1 An example of a 4-point function on a sphere.

Let us start with a map that has branch points appropriate for a 4-point correlation function of the form

$$\langle\sigma_n(0)\sigma_2(1)\sigma_2(w)\sigma_n(\infty)\rangle \tag{7.1}$$

Consider the map

$$z = Ct^n \frac{t-a}{t-1}, \tag{7.2}$$

where the parameter a will be related with coordinate w and the value of coefficient C will be determined below. The map (7.2) has two obvious ramification points: $z = 0$ and $z = \infty$, both of them give an n–th order branch point for nonzero values of a. For a general value of a the map (7.2) has two more ramification points; to find them we should look at the equation

$$\frac{dz}{dt} = 0. \tag{7.3}$$

For general value of a this equation reads:

$$t^{n-1}(nt^2 - t((n-1)a + (n+1)) + an) = 0. \tag{7.4}$$

The first factor corresponds to the obvious fact that at the point $t = 0$ we have a ramification point of n–th order, while the positions of the two "implicit" points of second order are given by:

$$t_{\pm} = \frac{1}{2n}\left((n-1)a + n + 1 \pm \sqrt{(a-1)((n-1)^2 a - (n+1)^2)}\right). \tag{7.5}$$

One of these points should correspond to $z = 1$; we let this be the point t_+. The other must correspond to $z = w$. By requiring $z(t_+) = 1$ we determine the value of coefficient C:

$$C = t_+^{-n} \frac{t_+ - 1}{t_+ - a}, \tag{7.6}$$

and we note that in what follows we will have

$$w = t_-. \tag{7.7}$$

Now we will analyze contributions to the Liouville action coming from the different ramification points. Let us start from the point $t = 0$. If $a \neq 0$ the map (7.2) near this point has the form:

$$z \approx Cat^n \tag{7.8}$$

and the inverse map is

$$t \approx \left(\frac{z}{aC}\right)^{1/n}. \tag{7.9}$$

The Liouville field and its derivative are given by:

$$\phi = \log\left(\frac{dz}{dt}\right) + c.c. \approx \log(nCat^{n-1}) + c.c., \qquad \partial_t\phi \approx \frac{n-1}{t}. \tag{7.10}$$

As usual we will cut a hole of radius ϵ around the point $z = 0$. Then the contribution to the Liouville action coming from the integration over the boundary of this hole is

$$S_L(t=0) = -\frac{n-1}{12}\log\left[n(aC)^{1/n}\epsilon^{\frac{n-1}{n}}\right]. \tag{7.11}$$

The same analysis near $t = \infty$ gives:

$$\begin{aligned} z \approx Ct^n, &\qquad t \approx \left(\frac{z}{C}\right)^{1/n}, \\ \phi \approx \log(nCt^{n-1}) + c.c., &\qquad \partial_t\phi \approx \frac{n-1}{t}, \\ S_L(t=\infty) = -(-1)\frac{n-1}{12}&\log\left[nC^{1/n}\tilde{\delta}^{\frac{1-n}{n}}\right]. \end{aligned} \tag{7.12}$$

Here we have cut a large circle of radius $1/\tilde{\delta}$ in z plane and the factor of (-1) in the last equation comes from the fact that we go around the point $t = \infty$ clockwise.

Near the point $t = t_-$ we get:

$$\begin{aligned} z \approx z_- + \xi_-(t-t_-)^2, &\qquad t - t_- \approx \left(\frac{z}{\xi_-}\right)^{1/2}, \\ \xi_- = \frac{1}{2}\left(\frac{d^2z}{dt^2}\right)_{t=t_-} &\approx \frac{nz_-(t_- - t_+)}{2t_-(t_- - a)(t_- - 1)}, \\ \phi \approx \frac{1}{2}\log(4(z-z_-)\xi_-) + c.c., &\qquad \partial_t\phi \approx \frac{1}{t}, \\ S_L(t=t_-) = -\frac{1}{24}&\log\left[4\epsilon\xi_-\right]. \end{aligned} \tag{7.13}$$

To get a contribution for $t = t_+$ one should make a replacement $+ \leftrightarrow -$ in the last expression; we also note that $z_+ = 1$. Thus we get:

$$\begin{aligned} S_L(t=t_+) &= -\frac{1}{24}\log\left[4\epsilon\xi_+\right], \\ \xi_+ &\approx \frac{n(t_+ - t_-)}{2t_+(t_+ - a)(t_+ - 1)}. \end{aligned} \tag{7.14}$$

Finally we should consider the images of $z = \infty$ that give finite values for t - there will be a puncture here corresponding to the boundary of the $|z|$ plane. As before we let this circle in the z plane have a radius $1/\delta$. The map (7.2) has only one such image: $t = 1$. The result is

$$
\begin{aligned}
z \approx C\frac{1-a}{t-1}, &\qquad t-1 \approx \frac{C(1-a)}{z}, \\
\phi \approx \log\left(-\frac{z^2}{C(1-a)}\right) + c.c., &\qquad \partial_t\phi \approx -\frac{2}{t-1} \\
S_L(t=1) = -\frac{1}{6}\log\left[C(1-a)\delta^2\right]. &
\end{aligned} \tag{7.15}
$$

Collecting all this information together and making obvious simplifications we finally get the expression for the Liouville action corresponding to our four point function:

$$
\begin{aligned}
S_L^{(1)} &= \left(-\frac{(n-1)^2}{12n} - \frac{1}{12}\right)\log\epsilon - \frac{(n-1)^2}{12n}\log\tilde{\delta} - \frac{1}{3}\log\delta \\
&- \frac{1}{4}\log C - \frac{1}{12}\log 2 - \frac{1}{12}\left(\frac{n+1}{2} - \frac{1}{n}\right)\log a - \frac{1}{8}\log(1-a) \\
&- \frac{1}{24}\log\left[(n-1)^2 a - (n+1)^2\right] - \frac{1}{12}\log n.
\end{aligned} \tag{7.16}
$$

The above expression gives the correlator $\langle\sigma_n^\epsilon(0)\sigma_2^\epsilon(1)\sigma_2^\epsilon(w)\sigma_n^{\tilde{\delta}}(\infty)\rangle$. To compute the correlation function of normalised twist operators, with one point at infinity, defined as

$$
\begin{aligned}
\langle\sigma_n(0)\sigma_2(1)\sigma_2(w)\sigma_n(\infty)\rangle &\equiv \lim_{|z|\to\infty}|z|^{4\Delta_n}\langle\sigma_n(0)\sigma_2(1)\sigma_2(w)\sigma_n(z)\rangle \\
&= \frac{\langle\sigma_n(0)\sigma_2(1)\sigma_2(w)\sigma_n^{\tilde{\delta}}(\infty)\rangle}{\langle\sigma_n(0)\sigma_n^{\tilde{\delta}}(\infty)\rangle}
\end{aligned} \tag{7.17}
$$

we use arguments similar to those in subsection 6.1. Then we find

$$
\begin{aligned}
\mathcal{F}_4 &\equiv \langle\sigma_n(0)\sigma_2(1)\sigma_2(w)\sigma_n(\infty)\rangle \\
&= \frac{\langle\sigma_n^\epsilon(0)\sigma_2^\epsilon(1)\sigma_2^\epsilon(w)\sigma_n^{\tilde{\delta}}(\infty)\rangle}{\langle\sigma_n^\epsilon(0)\sigma_n^{\tilde{\delta}}(\infty)\rangle}[\langle\sigma_2^\epsilon(0)\sigma_2^\epsilon(1)\rangle]^{-1}
\end{aligned} \tag{7.18}
$$

This leads to

$$
\begin{aligned}
\log\mathcal{F}_4 &= S_L^{(1)} - \log\left(\epsilon^{F_n}\tilde{\delta}^{F_n}\right) - \log\left(C_2\epsilon^{A_2}Q^{B_2}\right) + \frac{1}{3}(n+1-n)\log\delta \\
&+ ((n-1)-n)\log Q.
\end{aligned} \tag{7.19}
$$

where A_n, C_n, F_n are given in (3.19) and (6.13). The sourse of the last two terms is the fact that the numerator has $n+1$ fields transforming nontrivially, while the denominator has only n:

$$\begin{aligned} \langle\sigma_n^\epsilon(0)\sigma_n^{\tilde\delta}(\infty)\rangle &= \epsilon^{F_n}\tilde\delta^{F_n}e^{S_L^{(2)}+S_L^{(3)}}\frac{Z^{(\hat s)}}{(Z_\delta)^n}, & (7.20)\\ \langle\sigma_n^\epsilon(0)\sigma_2^\epsilon(1)\sigma_2^\epsilon(w)\sigma_n^\delta(\infty)\rangle &= e^{S_L^{(1)}}e^{S_L^{(2)}+S_L^{(3)}}\frac{Z^{(\hat s)}}{(Z_\delta)^{n+1}} & (7.21)\end{aligned}$$

We observe that the powers of regularization parameters cancel:

$$\begin{aligned} \log\epsilon: &\quad -\frac{(n-1)^2}{12n}-\frac{1}{12}-F_n-A_2=0,\\ \log\tilde\delta: &\quad -\frac{(n-1)^2}{12n}-F_n=0, \qquad \log\delta: \quad -\frac{1}{3}+\frac{1}{3}=0, \qquad (7.22)\\ \log Q: &\quad -n+(n-1)+(2-1)=0. \end{aligned}$$

This cancellation gives a consistency check on the calculations. The expression for the logarithm of the normalized four point function is given by:

$$\begin{aligned} \log\mathcal{F}_4 &= -\frac{1}{4}\log C-\frac{1}{12}\left(\frac{n+1}{2}-\frac{1}{n}\right)\log a-\frac{1}{8}\log(1-a)\\ &- \frac{1}{24}\log\left[(n-1)^2a-(n+1)^2\right]-\frac{1}{12}\log n-\frac{5}{12}\log 2. \quad (7.23)\end{aligned}$$

The value of C is given by (7.6).

7.2 Analysis of the 4-point function

Let us step back from the above calculation and think about the structure of a 4-point function $\langle\sigma_n(0)\sigma_2(1)\sigma_2(w)\sigma_n(\infty)\rangle$. Consider the limit $w\to 0$, and ask what operators are produced in the OPE of σ_n and σ_2. There are three possibilities, which must all be considered when we make the CFT operators O_j out of the sum over indices in the σ_j:

(a) The indices of σ_2 and the indices of σ_n have no overlap - i.e., the operators are of the form $\sigma_{12}\sigma_{34\ldots n+2}$. In this case the other two operators must have no overlapping indices either, and the entire 4-point function factors into two different parts $\langle\sigma_2\sigma_2\rangle\langle\sigma_n\sigma_n\rangle$. The covering surfaces are separate for the two parts, and we just multiply together the correlation functions obtained from the covering surfaces for the 2-point functions.

(b) The indices of σ_2 and the indices of σ_n have one overlap - i.e., the operators are of the form $\sigma_{12}\sigma_{23\ldots n+1}$. The OPE then produces the operator $\sigma_{123\ldots n+1} = \sigma_{n+1}$. The other two operators in the correlator must also have a singe overlap so that they can produce σ_{n+1}. The genus of the surface thus produced is seen to be

$$g = \frac{1+1+(n-1)+(n-1)}{2} - (n+1) + 1 = 0 \tag{7.24}$$

This case in fact corresponds to the surface that was constructed in the subsection above. Note that if we take σ_{12} around $\sigma_{23\ldots n+1}$ then it becomes σ_{13}. The OPE of σ_{13} with $\sigma_{23\ldots n+1}$ is still an operator of the form σ_{n+1}. On the other hand if we take the two σ_2 operators near each other, then we get the identity if we have $\sigma_{12}\sigma_{12}$, but we get σ_{123} if move the operators through a path such that they become σ_{12} and σ_{13}. In fact by moving the various operators around each other on the z plane, we can also get from the same correlator OPEs of the form $\sigma_{12}\sigma_{34}$ (which is nonsingular) and $\sigma_{12}\sigma_{n,n-1,\ldots 21}$ which produces an operator σ_{n-1}. Thus we should find singularities in the 4-point function arising from this surface to correspond to all these possibilities.

(c) The indices of σ_2 and the indices of σ_n have two overlaps, and the total number of indices involved in the correlator is $s = n$. (Note that case (b) above also could be brought to a form where σ_2 and σ_n have two overlaps, but the number of indices involved there overall was $n+1$.) The other two operators in the correlator must have a similar overlap of indices, since otherwise they cannot produce an operator that has only n distinct indices. In this case the genus of the covering surface is

$$g = \frac{1+1+(n-1)+(n-1)}{2} - n + 1 = 1. \tag{7.25}$$

We see that the correlator $\langle O_2 O_2 O_2 O_2\rangle$ will have contributions from correlators $\langle \sigma_2\sigma_2\sigma_2\sigma_2\rangle$ that give genus 0 and genus 1 surfaces, but no other surfaces. The genus 0 case is contained in the analysis in subsection 7.1, and we will study the genus 1 case in subsection 7.4 below. Note that for the genus 0 case we have many combinations of indices for the σ_2 operators as discussed in (b) above, but these all arise from different branches of the same function (7.23). We must thus add the results from these branches (as well as the disconnected part (case (a) above) and the genus 1 contribution) to obtain the complete 4-point function of the O_2 operators. We will not carry out the explicit addition since we expect the result to be simpler in the supersymmetric case, which we hope to present elsewhere.

7.3 Analysis of the $g = 0$ contribution

In this subsection, we analyse the correlator computed in (7.23), which corresponds to case (b) above, to check if it reproduces the expected short distance limits.

7.3.1 The limit $w \to 0$.

First let us consider the limit $w \to 0$, which corresponds to $t_- \to 0$ and $a \to 0$. For small values of a we have:

$$
\begin{aligned}
t_+ &\approx \frac{n+1}{n}, \qquad t_- \approx \frac{an}{n+1}, \qquad C \approx \frac{n^n}{(n+1)^{n+1}}, \\
z_- &\approx n^{2n}(n+1)^{-2-2n}a^{n+1}, \\
\log \mathcal{F}_4 &\approx \frac{1}{12}\left(\frac{1}{n(n+1)} - \frac{1}{2}\right)\log z_- - \frac{5}{12}\log 2 \\
&- \left(\frac{n}{6} + \frac{1}{6(n+1)} + \frac{1}{12}\right)\log n + \left(\frac{n}{6} + \frac{1}{6n} + \frac{1}{12}\right)\log(n+1).
\end{aligned} \tag{7.26}
$$

One can see that the correct singularity $(z_-)^{-2(\Delta_n+\Delta_2-\Delta_{n+1})}$ is reproduced. Using the expressions for three point functions we derived before one can check that in the limit $w \approx 0$:

$$
\log \mathcal{F}_4 \approx -2(\Delta_n + \Delta_2 - \Delta_{n+1})\log w + 2\log|C^{\sigma}_{n,2,n+1}|^2, \tag{7.27}
$$

which agrees with anticipated factorization.

7.3.2 The limit $w \to 1$

Let us now consider a limit $a \to 1$, which corresponds to one of two possible ways for point w to approach 1. After introducing $b = a - 1$ we get:

$$
\begin{aligned}
t_\pm &\approx 1 + \frac{b(n-1)}{2n} \pm \frac{1}{2n}\sqrt{-4nb}, \qquad C \approx 1, \\
t_\pm - a &\approx \pm\sqrt{-\frac{b}{n}} - \frac{b}{n}\frac{n+1}{2}, \\
z_- - z_+ &= \frac{z_-}{z_+} - 1 \approx \left(1 - 2\sqrt{-\frac{b}{n}}\right)^n \left(1 - (n-1)\sqrt{-\frac{b}{n}}\right)
\end{aligned}
$$

$$
\begin{aligned}
& \times \quad \left(1-(n+1)\sqrt{-\frac{b}{n}}\right)-1 \approx -4i\sqrt{nb}, \\
\log \mathcal{F}_4 \quad &\approx \quad -\frac{1}{4}\log(w-1) = -4\Delta_2 \log(w-1). \qquad (7.28)
\end{aligned}
$$

This singularity corresponds to σ_2 and σ_2 fusing to the identity.

There is another limit ($a \to \frac{(n+1)^2}{(n-1)^2}$) which also corresponds to $w \to 1$. Introducing $b = a-(n+1)^2/(n-1)^2$, we get:

$$
\begin{aligned}
t_\pm \quad &\approx \quad \frac{n+1}{n-1} \pm \sqrt{\frac{b}{n}}, \qquad C \approx -\left(\frac{n-1}{n+1}\right)^{n+1}, \\
\frac{dz}{dt} \quad &= \quad \frac{Cnt^n}{(t-1)^2}(t-t_+)(t-t_-) \approx -\frac{n(n-1)^4}{4(n+1)^2}(t-t_+)(t-t_-) \\
z_- - z_+ \quad &\approx \quad -\frac{n(n-1)^4}{4(n+1)^2}\int_{t_+}^{t_-} dt(t-t_+)(t-t_-) = -\frac{(n-1)^4 b^{3/2}}{3\sqrt{n}(n+1)^2}, \\
\log \mathcal{F}_4 \quad &\approx \quad -\frac{1}{36}\log(w-1) + \log(n-1)\left[\frac{1}{9}-\frac{1}{6}\left(n+\frac{1}{n}\right)\right] - \frac{2}{9}\log n \\
&+ \quad \log(n+1)\left[\frac{1}{6}\left(n+1+\frac{1}{n}\right)-\frac{1}{18}\right] - \frac{2}{3}\log 2 - \frac{1}{36}\log 3.
\end{aligned} \qquad (7.29)
$$

Using the equations (6.25) and (6.22) one can see that

$$
\log \mathcal{F}_4 \approx -2(\Delta_2+\Delta_2-\Delta_3)\log(w-1) + \log|C^\sigma_{223}|^2 + \log|C^\sigma_{n3n}|^2. \qquad (7.30)
$$

This corresponds to merging σ_2 and σ_2 to σ_3.

7.3.3 The limit $w \to \infty$.

We now look at the remaining limits of the expression (7.23) at which the four point function becomes singular. They emerge at the points where the coefficient C goes either to 0 or infinity, i.e. if the value of t_+ approaches one of the points: $0,\ 1,\ a,\ \infty$. Substituting this to the quadratic equation (7.4), we get the candidates for the critical values of a: $0,\ 1,\ \infty$. Two of these limits we already considered, now we analyze the last possibility: $a \to \infty$. In this limit we have:

$$
t_+ \approx a\frac{n-1}{n} - \frac{1}{n(n-1)}, \quad t_- \approx \frac{n}{n-1}. \qquad (7.31)
$$

So it is convenient to keep a point z_- fixed and vary the value of z_+ instead[2]. Thus we get:

$$\begin{aligned}
C &= t_-^{-n}\frac{t_- - 1}{t_- - a} \approx -\left(\frac{n-1}{n}\right)^n \frac{1}{a(n-1)}, \\
z_+ &\approx a^{n-1}\left(\frac{n-1}{n}\right)^{2n} \frac{1}{(n-1)^2}, \\
\log \mathcal{F}_4 &\approx \left(-\frac{1}{24} + \frac{1}{12n(n-1)}\right)\log z_+ + \log(n-1)\left[\frac{1}{12} - \frac{1}{6}\left(n + \frac{1}{n}\right)\right] \\
&+ \left[\frac{1}{6}\left(n - 1 + \frac{1}{n-1}\right) + \frac{1}{12}\right]\log n - \frac{5}{12}\log 2.
\end{aligned} \tag{7.32}$$

This expression can be rewritten in terms of three point functions:

$$\log \mathcal{F}_4 \approx 2(\Delta_n - \Delta_2 - \Delta_{n-1}) + 2\log |C^{\sigma}_{n-1,2,n}|^2, \tag{7.33}$$

thus it corresponds to the factorization of the following type:

$$\left\langle \left(\sigma_{(1\dots n)}\sigma_{(12)}\right)\left(\sigma_{(12)}\sigma_{(1\dots n)}\right)\right\rangle . \tag{7.34}$$

Thus the four point function reproduces the anticipated factorizations.

7.4 The $g = 1$ correlator $\langle\sigma_{12}\sigma_{12}\sigma_{12}\sigma_{12}\rangle$

Let us consider the case $n = 2$ in (7.1), so that we have the correlator

$$\langle\sigma_2(0)\sigma_2(1)\sigma_2(w)\sigma_2(\infty)\rangle \tag{7.35}$$

We wish to have the number of sheets over a generic point in the z plane to be 2; this gives $g = 1$ for the covering surface Σ. Each branch point is of order 2, so we seek a map of the form

$$\frac{dz}{dt} = \alpha[z(z-1)(z-w)(z-z_\infty)]^{1/2} \tag{7.36}$$

We choose not to put any branch point at infinity explicitly, the limit $z_\infty \to \infty$ will be taken in the end of the calculation. This equation may be solved

[2] Note that the definition of t_+ and t_- depend on the choice of branch for a multivalued function; in particular t_+ and t_- interchange if one goes along a small circle around the point $a = 1$.

using the Weierstrass function $\mathcal{P}$ and the solution in the $z_\infty \to \infty$ limit is given by:

$$z(t) = \frac{\mathcal{P}(t) - e_1}{e_2 - e_1} \tag{7.37}$$

where

$$\begin{aligned} e_1 &= \mathcal{P}(\frac{1}{2}), \quad e_2 = \mathcal{P}(\frac{\tau}{2}), \quad e_3 = \mathcal{P}(\frac{1}{2}+\frac{\tau}{2}) \\ w &= \left(\frac{\theta_3(\tau)}{\theta_4(\tau)}\right)^4 = \frac{e_3 - e_1}{e_2 - e_1} \end{aligned} \tag{7.38}$$

The coordinate t describes a torus given by modding out the complex plane with translations by 1 and τ.

We choose the fiducial metric on the torus to be that flat metric $d\hat{s}^2 = dtd\bar{t}$. Then we calculate the contribution to the Liouville action from the point $z = 0$. Near this point

$$\begin{aligned} \frac{dz}{dt} &\approx \alpha z^{1/2}\sqrt{-wz_\infty} \\ \phi &= \log\frac{dz}{dt} + c.c. \approx \log\left(\alpha z^{1/2}\sqrt{-wz_\infty}\right) + c.c. \\ \partial_t\phi &= \frac{dz}{dt}\partial_z\phi, \qquad \partial_z\phi \approx \frac{1}{2z} \end{aligned} \tag{7.39}$$

We can write $dt\partial_t\phi{=}dz\partial_z\phi$ for any infinitesimal segment of the contour of integration around the puncture, but we must circle the z plane puncture twice to circle the t space puncture once. We find it easier to work in the z plane instead of the t space to evaluate the integral around the puncture. We recall that the puncture has a radius ϵ in the z plane, and we put in a factor of 2 at the end to account for the relation between z and t contours. Then we get for the contribution to S_L:

$$S_L(z=0) = -\frac{1}{24}\log\left[\alpha^2\epsilon|wz_\infty|\right] \tag{7.40}$$

Similarly from the points $z = 1$, $z = w$ and $z = z_\infty$ we get the contributions

$$S_L(z=1) = -\frac{1}{24}\log\left[\alpha^2\epsilon|1-w||1-z_\infty|\right], \tag{7.41}$$

$$S_L(z=w) = -\frac{1}{24}\log\left[\alpha^2\epsilon|w||1-w||w-z_\infty|\right], \tag{7.42}$$

$$S_L(z=z_\infty) = -\frac{1}{24}\log\left[\alpha^2\epsilon|z_\infty||z_\infty||w-z_\infty|\right]. \tag{7.43}$$

Now we look at the image of infinity, where we have cut a circle $|z| = 1/\delta$. We have

$$\frac{dz}{dt} \approx \alpha z^2, \qquad \phi \approx \log[\alpha z^2] + c.c., \qquad \partial_z \phi \approx \frac{2}{z} \tag{7.44}$$

Noting that we must take the z plane contour clockwise, and putting in the factor of 2 to relate the contour to the t space contour, we get the contribution

$$S_L(z = \infty) = \frac{1}{3} \log \left[\alpha \delta^2\right] \tag{7.45}$$

Adding up all contributions we get

$$\begin{aligned} S_L^{(1)} &= -\frac{1}{6} \log \epsilon - \frac{2}{3} \log \delta - \frac{1}{12} \log |w(1-w) z_\infty (z_\infty - 1)(z_\infty - w)| \\ &\approx -\frac{1}{6} \log \epsilon - \frac{2}{3} \log \delta - \frac{1}{12} \log |w(1-w)| - \frac{1}{4} \log |z_\infty| \end{aligned} \tag{7.46}$$

Note that α does not appear in the final result, as one could anticipate from the fact that this constant can be absorbed in the rescaling of the t plane. In the case under consideration there is no contribution to the Liouville action coming from the $|z| > 1/\delta$ region: there is no curvature on the torus t and

$$\frac{d\tilde{z}}{dt} = -\frac{1}{\delta^2 z^2} \frac{dz}{dt} \approx -\frac{\alpha}{\delta^2}, \tag{7.47}$$

giving a constant ϕ to leading order and thus a vanishing kinetic term for ϕ. Thus $S_L = S_L^{(1)}$ and the general expression (2.44) gives:

$$\langle \sigma_2(0) \sigma_2(1) \sigma_2(w) \sigma_2(z_\infty) \rangle_\delta = e^{S_L^{(1)}} \frac{Z^{(\hat{s})}}{(Z_\delta)^2}. \tag{7.48}$$

To obtain the normalized 4-point function we write

$$\begin{aligned} \langle \sigma_2(0) \sigma_2(1) \sigma_2(w) \sigma_2(\infty) \rangle &\equiv \lim_{|z_\infty| \to \infty} |z_\infty|^{4\Delta_2} \langle \sigma_2(0) \sigma_2(1) \sigma_2(w) \sigma_2(z_\infty) \rangle \\ &= \lim_{|z_\infty| \to \infty} |z_\infty|^{4\Delta_2} \frac{\langle \sigma_2^\epsilon(0) \sigma_2^\epsilon(1) \sigma_2^\epsilon(w) \sigma_2^{\tilde{\delta}}(z_\infty) \rangle}{\langle \sigma_2^\epsilon(0) \sigma_2^\epsilon(1) \rangle^2} \\ &= 2^{-2/3} |w(1-w)|^{-\frac{1}{12}} Z_\tau \end{aligned} \tag{7.49}$$

Here we have used the fact that in this case the partition function $Z^{(\hat{s})}$ in (2.16) is that on the flat torus with modular parameter τ given through (7.38).

Since the group S^2 equals the group Z_2, we can compare (7.49) with the 4-point function obtained for σ_2 operators for the Z_2 orbifold in [9]. (7.49) agrees with (4.13) of [9] for the case of a noncompact boson field and with (4.16) for the compact boson field.

One observes that if the fields X^i are noncompact bosons, then as $w \to 1$ we find a factor $\log(w-1)$ in the OPE in addition to the expected power $(w-1)^{1/4}$. We suggest the following interpretation of this logarithm. There is a continuous family of momentum modes for the noncompact boson, with energy going to zero. If we do not orbifold the target space, then momentum conservation allows only a definite momentum mode to appear in the OPE of two fields. But the orbifolding destroys the translation invariance in $X^1 - X^2$, and nonzero momentum modes can be exchanged between sets of operators where each set does not carry any net momentum charge. The exchange of such modes (with dimensions accumulating to zero) between the pair $\sigma_2(w)\sigma_2(1)$ and the pair $\sigma_2(0)\sigma_2(\infty)$ gives rise to the logarithm. Of course when the boson is compact, this logarithm disappears, as can be verified from (7.49) or the equivalent results in [9].

8 Discussion

The motivation for our study of correlation functions of symmetric orbifolds was the fact that the dual of the $AdS_3 \times S^3 \times M$ spacetime (which arises in black hole studies) is the CFT arising from the low energy limit of the D1-D5 system, and the D1-D5 system is believed to be a deformation of an orbifold CFT (with the undeformed orbifold as a special point in moduli space). To study this duality we must really study the supersymmetric orbifold theory, while in this paper we have just studied the bosonic theory. It turns out however that the supersymmetric orbifold can be studied with only a small extension of what we have done here. Following [14] we can bosonize the fermions. Then if we go to the covering space Σ near the insertion of twist operator then we find only the following difference from the bosonic case – at the location of the twist operator we do not have the identity operator, but instead a 'charge operator' of the form $P(\partial X_a, \partial^2 X_a, \ldots) e^{i \sum k_a X_a}$. Here X_a are the bosons that arise from bosonizing the complex fermions, and P is a polynomial expression in its arguments. It is easy to compute the correlation function of these charge operators on the covering space Σ, and then we have the twist correlation functions for the supersymmetric theory. We will

present this calculation elsewhere, but here we note that many properties of interest for the supersymmetric correlation functions can be already seen from the bosonic analysis that we have done here. In this section we recall the features of the AdS/CFT duality map and analyse some properties of the 3-point functions in the CFT.

8.1 'Universality' of the correlation functions

We have mentioned before that while we have discsussed the orbifold theory R^N/S_N (where the coordinate of R gave X, a real scalar field), we could replace the CFT of X by any other CFT of our choice, and the calculations performed here would remain essentially the same. When the covering surface Σ had genus zero, the results depend only on the value of c, and thus if we had $(T^4)^N/S_N$ theory or a $(K_3)^N/S_N$ theory, then we would simply choose $c = 4$ in the Liouville action (2.17) (instead of $c = 1$). If Σ had $g = 1$, then we would need to put in the partition function (of a single copy) of T^4 or K_3 for the value of $Z^{(\hat{s})}$ in (2.16). But apart from the value of c and the value of partition functions on Σ there is no change in the calculation. Thus in particular the 3-point functions that we have computed at genus zero are universal in the sense that if we take them to the power c then we get the 3-point functions for any CFT of the form M^N/S_N with the CFT on M having cenral charge c.

There is a small change in the calculation when we consider the supersymmetric case. The fermions from different copies of M anticommute, and the twist operators carry a representation of the R symmetry. As a consequence the dimensions of the twist operators are not given by (3.21), but for the supersymmetric theory based on $M = T^4$ are given by $\frac{1}{2}(n-1)$. However as mentioned above, our analysis can be extended with small modifications to such theories as well.

Note that our method does not work if we have an orbifold group other than S_N. Thus for example if we had a Z_N orbifold of a complex boson [9], then we could go to the covering space over a twist operator σ_n, but not write the CFT in terms of an unconstrained field on this covering space. The reason is that we have n sheets or more of the cover over any point in the base space, but the central charge of the theory is just 2, and so we cannot attribute one scalar field to each sheet of the cover. Thus our method, and its associated universalities, are special to S_N orbifolds, where a twist operator just permutes copies of a given CFT but does not exploit any special

symmetry of the CFT itself.

8.2 The genus expansion and the fusion rules of WZW models

We have studied the orbifold CFT on the plane, but found that the correlation functions can be organized in a genus expansion, arising from the genus of the covering surface Σ. In the large N limit the contribution of a higher genus surface goes like $1/N^{g+\frac{1}{2}}$. This situation is similar to that in the Yang-Mills theory that is dual to $AdS_5 \times S^5$. The Yang-Mills theory has correlation functions that can be expanded in a genus expansion, with higher genus surfaces supressed by $1/N^g$. In the Yang-Mills theory the genus expansion has its origins in the structure of Feynman diagrams for fields carring two indices (the 'double line representation' of gauge bosons). In our case we have quite a different origin for the genus expansion. In the case of $AdS_5 \times S^5$ it is believed that the genus expansion of the dual Yang-Mills theory is related to the genus expansion of string theory on this spacetime, though the precise relationship is not clear. It would be interesting if the genus expansion we have for the $D1-D5$ CFT would be related to the genus expansion of the string theory on $AdS_3 \times S^3 \times M$.

In this context we observe the following relation. It was argued in [7] that the orbifold CFT M^N/S_N indeed corresponds to a point in the D1-D5 system moduli space. Further, at this point we have the number of 1-branes (n_1) and of 5-branes (n_5) given by $n_5 = 1, N = n_1 n_5 = n_1$. The dual string theory is in general an $SU(2)$ Wess-Zumino-Witten (WZW) model [18], though at the orbifold point of the CFT this string theory is complicated to analyze. The twist operators σ_n, $n = 1 \ldots N$ of the CFT (σ_1 =Identity) correspond to WZW primaries with $j = (n-1)/2,\ 0 \le j \le \frac{N-1}{2}$. Since in a usual WZW model we have $0 \le j \le k/2$, we set $k = N - 1$.

The fusion rules for the WZW model, which give the 3-point functions of the string theory on the sphere (tree level) are as follows. The spins j follow the rules for spin addition in $SU(2)$, except that there is also a 'truncation from above'

$$\begin{array}{rcl} (j_1, j_2) & \to & j_3 \\ |j_1 - j_2| & \le & j_3 \ \le \ |j_1 + j_2|, \qquad j_1 + j_2 + j_3 \ \le \ k \end{array} \tag{8.1}$$

Now consider the 3-point function in the orbifold CFT, for the case where

the genus of the covering surface Σ is $g = 0$. The ramification order of Σ at the insertion of σ_{n_i} is $r_i = (n_i - 1) = 2j_i$. The rules in (4.4), (4.5), (4.6) translate to $|j_1 - j_2| \le j_3 \ \le \ |j_1 + j_2|$. Further, the number of sheets s is bounded as $s \le N$. Then the relation (4.1) gives

$$\sum_i \frac{r_i}{2} = g - 1 + s \le -1 + N \quad \rightarrow \quad j_1 + j_2 + j_3 \le k \tag{8.2}$$

While (8.2) is a relation for the bosonic orbifold theory, we expect an essentially similar relation for the supersymmetric case. Thus we observe a similarity between the $g = 0$ 3-point functions of the WZW model (8.1) and of the CFT (8.2).

At genus $g = 1$ however, we find that any three spins j_1, j_2, j_3 can give a nonzero 3-point function in the string theory. In the orbifold CFT, however, we get only a slight relaxation of the rule (8.2): we get $j_1 + j_2 + j_3 \le k + 1$. Roughly speaking we can reproduce this rule in the string theory if we require that in the string theory one loop diagram there be a way to draw the lines such that only spins $j \le 1/2$ be allowed to circulate in the loop. Of course we are outside the domain of any good perturbation expansion at this point, since if the spins are of order k then there is no small parameter in the theory to expand in, and thus there is no requirement that there be an exact relation between the rules in a WZW string theory and the rules in the orbifold CFT.

We note that in [14] the 3-point functions of chiral primaries that were studied had 'one overlap' in their indices. This corresponds to $j_1 + j_2 = j_3$ in the above fusion rules, and since for the supersymmetric case the dimension is linear in the charge, we also have $\Delta_1 + \Delta_2 = \Delta_3$. This corresponds to the case of 'extremal' correlation functions in the language of [20]. In [14] the 3-point correlators for this special case were found by an elegant recursion relation, which arises from the fact that there is no singularity in the OPE, and thus the duality relation of conformal blocks becomes a 'chiral ring' type of associativity law among the fusion coefficients. It is not clear however how to extend this method to the non-extremal case, and one motivation for the present work was to develop a scheme to compute the correlators for $j_1 + j_2 < j_3$, which corresponds to more than one overlap. In the case of one overlap we have extended our calculations to the supersymmetric case, and found results in agreement with [14].

8.3 3-point couplings and the stringy exclusion principle.

In the $AdS_5 \times S^5$ case the 3-point couplings of supergravity agree with the large N limit of the 3-point functions in the free Yang-Mills theory; thus there is a nonrenormalization of this correlator as the coupling g is varied. It is not clear if a similar result holds for the $AdS_3 \times S^3 \times M$ case, and even less clear what nonrenormalization theorems hold at finite N. But it is nevertheless interesting to ask how the correlators in the orbifold CFT behave as we go from infinite N to finite N, and in particular what happens as we approach the limits of the stringy exclusion principle.

Thus we examine the ratio

$$R(m,n,q;\bar{N}) \equiv \frac{\sqrt{\bar{N}}\langle O_n O_m O_q\rangle_{\bar{N}}}{\lim_{N\to\infty}\sqrt{N}\langle O_n O_m O_q\rangle_N} \tag{8.3}$$

where the subscripts on the correlator give the value of N. We have rescaled the correlators by $\sqrt{N}$ to obtain the effective coupling of the 3-point function; the correlator itself goes as $1/\sqrt{N}$. For $n,m,q << \bar{N}$ we expect $R \approx 1$, while as n,m,q become order $\bar{N}$ we expect that R will fall to zero. We take the case of the 3-point function with single overlap, and further set $m = n$. Then we have $q = 2n-1$, and we write

$$R(n,n,2n-1;\bar{N}) \equiv R(n;\bar{N}) \tag{8.4}$$

It is easy to see that for the case of single overlap the correlators $\langle \sigma_n \sigma_m \sigma_{m+n-1}\rangle$ can get a contribution only from surfaces Σ with $g = 0$, for which case we have done a complete calculation of the correlator and its combinatorics. Note further that in the ratio (8.3) the actual value of $\langle \sigma_n \sigma_m \sigma_{m+n-1}\rangle$ will cancel, and the value of R will be determined by combinatorial factors. These factors are expected to be the same for the bosonic and supersymmetric cases.

In the figure we plot $R(n;\bar{N})$ versus n (for $\bar{N} = 1000$). We see that R drops significantly after n exceeds $\sim \sqrt{\bar{N}}$. This effect can be traced to the fact that the number of ways to select s ordered indices from $\bar{N}$ indices is

$$\begin{aligned}\bar{N}(\bar{N}-1) \quad \ldots \quad (\bar{N}-s+1) &= \bar{N}^s(1-\frac{1}{\bar{N}})(1-\frac{2}{\bar{N}})\ldots(1-\frac{s-1}{\bar{N}}) \\ \approx \quad \bar{N}^s(1-\frac{1}{\bar{N}}\sum_{j=1}^{s-1} j) &= \bar{N}^s(1-\frac{1}{\bar{N}}\frac{s(s-1)}{2})\end{aligned} \tag{8.5}$$

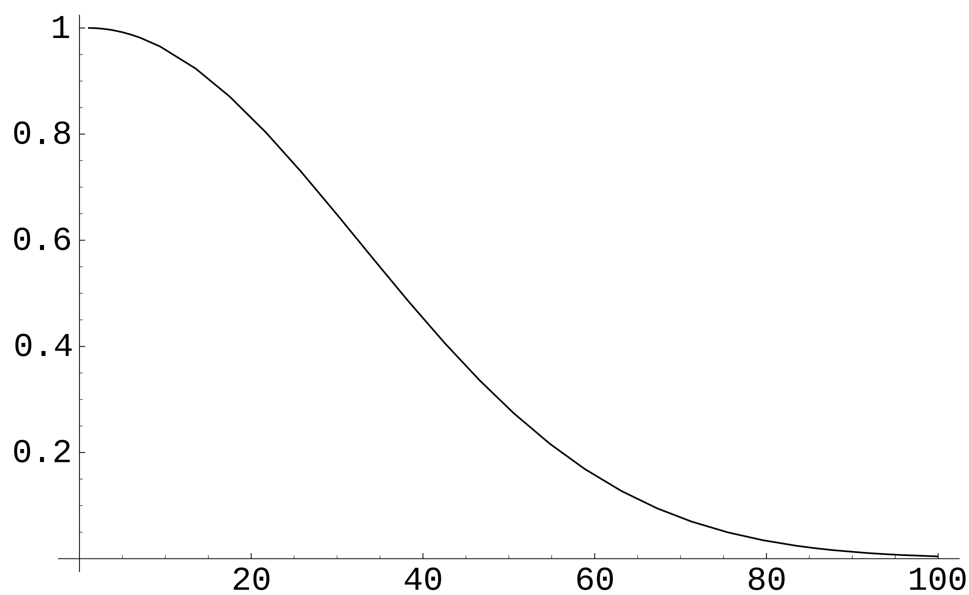

Figure 1: The factor (8.4) as function of n for $\bar{N} = 1000$.

If the CFT 3-point function is indeed not renormalized for finite N, then the above result has interesting implications. The coupling between three gravitons would then be a constant for low energies ($n << \bar{N}$) but would drop rapidly for high energies. Thus the behavior of high frequency modes would not follow a naive 'equivalence principle'.

This issue may be relevant to Hawking's derivation of black hole radiation, where we need to make a change of coordinates to study the high frequency modes near the horizon. For these modes to evolve as used in the derivation, we use implicitly the naive value of the following cubic coupling: that of a low energy graviton (representing the attraction of the hole) and two high energy quanta (representing the high energy mode emerging from the horizon, getting redshifted by the attraction of the hole). If this coupling differs from the one expected from naive gravitational physics, then the semiclassical derivation of Hawking radiation may require modification, with corresponding implications for the information paradox.

8.4 Conclusion

It would be important to pursue further the study of the supersymmetric case, and to compare with the dual superstring theory. The subset of correlators computed in [14] was compared to supergravity in [15], but it was a little unclear how closely the two calculations agreed. A better picture may emerge when we look at the complete set of correlators of the supersymmetric side, which is possible to do by extending our computation here to include the R-charges carried by the twist operators in the supersymmetric case.

It was argued recently in [21] that the CFT of the D1-D5 system exhibits a duality to a set of spacetimes, of which the AdS space is only one member. If the 3-point functions are protected against coupling changes then we should see a reflection of this fact in correlators at the orbifold point.

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